

# GENERALIZED METRIC PROPERTIES OF SPHERES AND RENORMING OF NORMED SPACES

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**ABSTRACT.** We study some generalized metric properties of weak topologies when restricted to the unit sphere of some equivalent norm on a Banach space, and their relationships with other geometrical properties of norms. In case of dual Banach space  $X^*$ , we prove that there exists a dual norm such that its unit sphere is a Moore space for the weak\*-topology (has a  $G_\delta$ -diagonal for the weak\*-topology, respectively) if, and only if,  $X^*$  admits an equivalent weak\*-LUR dual norm (rotund dual norm, respectively).

## 1. INTRODUCTION

Throughout this paper  $X$  will denote a normed space and  $X^*$  will denote its topological dual. If  $F$  is a subset of  $X^*$ , then  $\sigma(X, F)$  denotes the weakest topology on  $X$  that makes each member of  $F$  continuous or, equivalently, the topology of pointwise convergence on  $F$ . Analogously, if  $E$  is a subset of  $X$ , then  $\sigma(X^*, E)$  is the topology for  $X^*$  of pointwise convergence on  $E$ . We denote with  $\mathcal{B}_X$  (respectively,  $\mathcal{S}_X$ ) the unit ball of  $X$  (respectively, the unit sphere of  $X$ ). Recall that a subset  $B$  of  $\mathcal{B}_{X^*}$  is said to be *norming* if

$$\|x\|_B = \sup_{b^* \in B} |b^*(x)|$$

is a norm on  $X$  equivalent to the original norm of  $X$ . Observe that the definition of  $\|\cdot\|_B$  is plenty of sense also for element  $x^{**}$  in the bidual space  $X^{**}$ . A subspace  $F \subseteq X^*$  is norming if  $F \cap \mathcal{B}_{X^*}$  is norming. Finally, when  $C$  is a convex set of a vector space, we will denote with  $\text{ext}(C)$  the set of extreme point of  $C$ .

A norm on a normed space  $X$  is said to be *rotund* (or *strictly convex*) if, given  $x, y \in X$  satisfying  $\|x\| = \|y\| = \|\frac{1}{2}(x+y)\|$  (equivalently,  $2\|x\|^2 + 2\|y\|^2 - \|x+y\|^2 = 0$ ), we have  $x = y$  (see [Cla36, pag. 404]). Geometrically, this means that the unit sphere  $\mathcal{S}_X$  of  $X$  in this norm has no non-trivial line segment, or equivalently  $\text{ext}(\mathcal{B}_X) = \mathcal{S}_X$ .

Let  $\tau$  be a topology on a normed space  $X$ .  $X$  is said to be  $\tau$ -locally uniformly rotund ( $\tau$ -LUR, for short) if given  $x \in X$  and  $(x_n)_{n \in \mathbb{N}} \subseteq X$ , then  $x_n$  converges to  $x$  in the  $\tau$ -topology, whenever

$$\lim_{n \rightarrow +\infty} (2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2) = 0.$$

If  $\tau$  is the norm topology, we simply say that  $X$  is LUR.

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*Date:* May 27, 2016

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2010 *Mathematics Subject Classification.* 46B03, 46B20, 46B26, 54E35.

*Key words and phrases.* Generalized metric spaces, renorming, Banach spaces, Moore spaces,  $G_\delta$ -diagonal, symmetric, semimetrics, slice derivation, metric spaces, LUR norm, rotund norm, strictly convex norm.

There are clearly many normed space whose natural norms are not rotund. However by using topological and linear property of normed space is sometimes possible to define an equivalent rotund norm. In certain cases, we would like the new norm to possess some form of lower semicontinuity. For instance we may wish for a norm on a dual Banach space  $X^*$  to be  $w^*$ -lower semicontinuous, so that it is the dual of some norm on  $X$ . Obviously such condition can make the norms more difficult to construct, but we can obtain some benefits. For example, if  $X^*$  admits an equivalent dual rotund norm, then  $X$  admits an equivalent Gâteaux smooth norm (see [Šmu40] and [DGZ93]).

Despite the simple nature of rotundness, the question of whether a normed space admits an equivalent rotund norm is actually rather difficult to answer in general. For a full account of all the known results the reader can refer to [DGZ93], [God01], [Ziz03] and [ST10]. One recent result is the characterization contained in [OST12], in order to state it here we need the following definition (see [OST12, Definition 2.6]):

**Definition 1.1** ((\*)-property). A topological space  $A$  has *property (\*)* if, and only if, it admits a sequence  $(\mathcal{U}_n)_{n \in \mathbb{N}}$  of families of open sets such that for every  $x, y \in A$  there exists  $n_0 \in \mathbb{N}$  such that  $\mathcal{U}_{n_0}$  (\*)-separates  $x$  and  $y$ , i.e.

- (1)  $\{x, y\} \cap \bigcup \mathcal{U}_{n_0} \neq \emptyset$ , where  $\bigcup \mathcal{U}_{n_0} := \bigcup \{U \mid U \in \mathcal{U}_{n_0}\}$ ;
- (2) for every  $U \in \mathcal{U}_{n_0}$  the set  $\{x, y\} \cap U$  is at most a singleton.

We will call  $(\mathcal{U}_n)_{n \in \mathbb{N}}$  a *(\*)-sequence* for  $A$ . If  $A$  is a subset of a topological vector space and every family  $\mathcal{U}_n$  is formed by open slices of  $A$ , then we say that  $A$  has *(\*) with slices*.

This is a sort of separation property for point in a topological space. This definition is a generalization of the notion of  $G_\delta$ -diagonal property (see [Gru84, Section 2] and definition 4.1). We state the main result of [OST12] in the following theorem (see [OST12, Theorem 2.7]):

**Theorem 1.2.** *Let  $X$  be a normed space and  $F$  in  $X^*$  a norming subspace. The following facts are equivalent:*

- (1)  $X$  admits an equivalent,  $\sigma(X, F)$ -lower semicontinuous and rotund norm;
- (2)  $(X, \sigma(X, F))$  has *(\*) with slices*;
- (3)  $(\mathcal{S}_X, \sigma(X, F))$  has *(\*) with slices*.

In order to prove such a result the authors of [OST12] used a slice localization theorem already appeared in [OT09b, Theorem 3].

**Theorem 1.3** (Slice localization theorem). *Let  $X$  be a normed space with a norming subspace  $F$  in  $X^*$ . Let  $A$  be a bounded subset in  $X$  and  $\mathcal{H}$  a family of  $\sigma(X, F)$ -open half-spaces such that for every  $H \in \mathcal{H}$  the set  $A \cap H$  is nonempty. There exists an equivalent  $\sigma(X, F)$ -lower semicontinuous norm  $\|\cdot\|_{A, \mathcal{H}}$  such that for every sequence  $(x_n)_{n \in \mathbb{N}} \subseteq X$  and  $x \in A \cap H$  for some  $H \in \mathcal{H}$ , if*

$$\lim_{n \in \mathbb{N}} (2\|x\|_{A, \mathcal{H}}^2 + 2\|x_n\|_{A, \mathcal{H}}^2 - \|x + x_n\|_{A, \mathcal{H}}^2) = 0,$$

*then there exists a sequence of  $\sigma(X, F)$ -open half-spaces  $\{H_n\}_{n \in \mathbb{N}} \subseteq \mathcal{H}$  such that*

- (1) *there exists  $n_0 \in \mathbb{N}$  such that  $x, x_n \in H_n$  for  $n \geq n_0$ , if  $x_n \in A$ ;*
- (2) *for every  $\delta > 0$  there exists  $n_\delta \in \mathbb{N}$  such that*

$$x, x_n \in \overline{\text{conv}(A \cap H_n) + \delta \mathcal{B}_X}^{\sigma(X, F)},$$

for every  $n \geq n_\delta$ .

Analysing the construction it is possible to assume that every one of the elements of the sequence  $(H_n)_{n \in \mathbb{N}}$  contains the point  $x$ . The aim of the present paper is to prove a stronger version of Theorem 1.3, namely Theorem 1.4, and show how these results can be applied to characterize the existence of equivalent norm on normed spaces, whenever there exists an equivalent norm such that its sphere has some generalized metric properties (see [Gru84]).

This paper is organized as follows. In section 2 we will prove a “stronger” version of Theorem 1.3, namely

**Theorem 1.4** (Strong connection lemma). *Let  $X$  a normed space and  $F$  a norming subspace in  $X^*$ . Let  $A$  a bounded subset in  $X$  and  $\mathcal{H}$  a family of  $\sigma(X, F)$ -open halfspace such that for every  $H \in \mathcal{H}$  the set  $A \cap H$  is nonempty. A  $\sigma(X, F)$ -lower semicontinuous norm  $\|\cdot\|_{A, \mathcal{H}}$  exists with the following property: for every  $x \in X$  and  $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \subseteq X$ , if  $x \in A \cap \bigcup \mathcal{H}$  and*

$$\begin{aligned} \lim_{n \rightarrow +\infty} (2\|x\|_{A, \mathcal{H}}^2 + 2\|x_n\|_{A, \mathcal{H}}^2 - \|x + x_n\|_{A, \mathcal{H}}^2) &= 0; \\ \lim_{n \rightarrow +\infty} (2\|x_n\|_{A, \mathcal{H}}^2 + 2\|y_n\|_{A, \mathcal{H}}^2 - \|x_n + y_n\|_{A, \mathcal{H}}^2) &= 0, \end{aligned}$$

then a sequence  $(H_n^L)_{n \in \mathbb{N}} \subseteq \mathcal{H}$  exists with the following properties:

- (1) if  $x_n \in A$  eventually, then there exists  $n_L \in \mathbb{N}$  such that  $x, x_n \in H_n^L$  for  $n \geq n_L$ ;
- (2) for every  $\delta > 0$  there is some  $n_{L, \delta} \in \mathbb{N}$  such that  $x \in H_n^L$  and

$$x_n \in \text{conv}(A \cap H_n^L) + \delta \mathcal{B}_X \quad \text{for } n \geq n_{L, \delta}.$$

Moreover, a sequence  $(H_n^U)_{n \in \mathbb{N}} \subseteq \mathcal{H}$  exists with the following property: if an increasing sequence  $n_1 < n_2 < \dots < n_k < \dots$  and  $k_0 \in \mathbb{N}$  exist such that

- (3) both  $x_{n_k} \in A \cap \bigcup \mathcal{H}$  and  $y_{n_k} \in A \cap \bigcup \mathcal{H}$  for  $k \geq k_0$ , then  $k_U \in \mathbb{N}$  exists with

$$x_{n_k}, y_{n_k} \in H_{n_k}^U \quad \text{for } k \geq k_U;$$

- (4) either  $x_{n_k} \in A \cap \bigcup \mathcal{H}$  or  $y_{n_k} \in A \cap \bigcup \mathcal{H}$  for  $k \geq k_0$ , then for every  $\delta > 0$  there exists  $k_{U, \delta} \in \mathbb{N}$  with

$$x_{n_k}, y_{n_k} \in \text{conv}(A \cap H_{n_k}^U) + \delta \mathcal{B}_X \quad \text{for } k \geq k_{U, \delta}.$$

Furthermore, if  $x_{n_k} \in A \cap \bigcup \mathcal{H}$  for  $k \geq k_0$ , then  $x_{n_k} \in H_{n_k}^U$  for  $k \geq k_{U, \delta}$ .

This theorem will be important for give an alternative proof of some results of [FOR16]. Section 3 is devoted to introduce a new slice derivation process with the idea of replace open slices with open neighbourhood in the topological construction done in order to obtain an equivalent rotund norm.

We recall that a topological space  $X$  is said to have a  $G_\delta$ -diagonal if, and only if, there exists a sequence  $(\mathcal{G}_n)_{n \in \mathbb{N}}$  of open covers of  $X$  such that for each  $x, y \in X$  with  $x \neq y$ , there exists  $n \in \mathbb{N}$  with

$$y \notin \text{st}(x, \mathcal{G}_n) := \bigcup \{U \in \mathcal{G}_n \mid x \in U\}.$$

In addition, if  $X$  is a subset of a topological vector space and  $(\mathcal{G}_n)_{n \in \mathbb{N}}$  a  $G_\delta$ -diagonal sequence for  $X$ , with the property that every element of  $\bigcup_{n \in \mathbb{N}} \mathcal{G}_n$  is a  $\sigma(X, F)$ -open slice of  $X$ , we

say that  $X$  has a  $G_\delta$ -diagonal with  $\sigma(X, F)$ -slices. In Section 4 we will prove the following result.

**Theorem 1.5.** *Let  $X$  be a normed space and  $F \subseteq X^*$  a norming subspace. The following facts are equivalent*

- (1)  $X$  admits an equivalent,  $\sigma(X, F)$ -lower semicontinuous and rotund norm  $\|\cdot\|_R$ ;
- (2)  $X$  admits an equivalent,  $\sigma(X, F)$ -lower semicontinuous norm  $\|\cdot\|_\delta$  such that its unit sphere has a  $G_\delta$ -diagonal for the  $\sigma(X, F)$ -topology, with  $\sigma(X, F)$ -slices.

If  $X$  is a dual Banach space  $Y^*$  and  $F = Y$ , then (1) is equivalent to

- (3)  $Y^*$  admits an equivalent dual norm  $\|\cdot\|_G$  such that its unit sphere has a  $G_\delta$ -diagonal for the weak\*-topology.

We remind the reader that if  $S$  is a non empty set then a function  $\rho : S \times S \rightarrow [0, +\infty)$  is said to be a symmetric if  $\rho(x, y) = \rho(y, x)$  for every  $x, y \in S$ , and  $\rho(x, y) = 0$  if, and only if,  $x = y$ . In section 5 we will generalize a classical result by S. Troyanski (see [Tro79] and [DGZ93]), i.e. a Banach space  $X$  admits an equivalent LUR norm if, and only if, there exists an equivalent norm  $\|\cdot\|_D$  such that every point of its unit sphere is denting (for every  $\varepsilon > 0$  and  $x \in \mathcal{S}_D$  there exists a  $w$ -open half-space  $H$  such that  $x \in H$  and  $\|\cdot\|_D$ -diam( $H \cap \mathcal{B}_D$ )  $< \varepsilon$ ). In particular we will prove the following result.

**Theorem 1.6.** *Let  $X$  be a normed space,  $F \subseteq X^*$  a norming subspace and  $S$  a nonempty set with a symmetric  $\rho$ . Let  $\Phi : X \rightarrow S$  be a map such that for every  $x \in \mathcal{S}_X$  and every  $\varepsilon > 0$  there exists a  $\sigma(X, F)$ -open half-space  $H$  with  $x \in H$  and*

$$\rho\text{-diam}(\Phi(H \cap \mathcal{B}_X)) < \varepsilon.$$

*Then there exists a  $\sigma(X, F)$ -lower semicontinuous norm  $\|\cdot\|_\Phi$  such that*

$$\lim_{n \rightarrow +\infty} \rho(\Phi(x_n), \Phi(x)) = 0,$$

*whenever  $x \in \mathcal{S}_X$ ,  $(x_n)_{n \in \mathbb{N}} \subseteq \mathcal{B}_X$  and  $2\|x\|_\Phi^2 + 2\|x_n\|_\Phi^2 - \|x + x_n\|_\Phi^2 \rightarrow 0$ . Furthermore if  $2\|x\|_\Phi^2 + 2\|y\|_\Phi^2 - \|x + y\|_\Phi^2 = 0$ , for some point  $x \in \mathcal{S}_X$  and  $y \in \mathcal{B}_X$ , then  $\Phi(x) = \Phi(y)$ .*

We recall that a regular topological space  $X$  is a Moore space if there exists a sequence  $(\mathcal{G}_n)_{n \in \mathbb{N}}$  of open covers of  $X$  such that for each  $x \in X$ ,

$$\{\text{st}(x, \mathcal{G}_n) \mid n \in \mathbb{N}\}$$

is a neighbourhood base at  $x$ . In addition if  $X$  is a subset of a topological vector space and  $(\mathcal{G}_n)_{n \in \mathbb{N}}$  a development of  $X$ , with the property that every element of  $\bigcup_{n \in \mathbb{N}} \mathcal{G}_n$  is a  $\sigma(X, F)$ -open slice of  $X$ , we say that  $X$  is a Moore space with  $\sigma(X, F)$ -slices. In Section 6 we prove the following result.

**Theorem 1.7.** *Let  $X$  a normed space and  $F$  a norming subspace in  $X^*$ . The following facts are equivalent:*

- (1)  $X$  admits an equivalent  $\sigma(X, F)$ -lower semicontinuous and  $\sigma(X, F)$ -LUR norm;
- (2)  $X$  admits an equivalent  $\sigma(X, F)$ -lower semicontinuous norm  $\|\cdot\|_M$  such that its unit sphere is a Moore space with  $\sigma(X, F)$ -slices, for the  $\sigma(X, F)$ -topology.

If  $X$  is a dual Banach space  $Y^*$  and  $F = Y$ , then (1) is equivalent to

- (3)  $Y^*$  admits an equivalent dual norm  $\|\cdot\|_*$  such that its unit sphere is a Moore space for the weak\*-topology.

## 2. A STRONG CONNECTION LEMMA

In this section we will recall some basic results that will be used in the paper. Moreover we will prove Theorem 1.4. The following lemma is well known, but prove to be useful in our computations (see [FOR16]).

**Lemma 2.1.** *Let  $f$  be a real valued convex function on a normed space  $X$ . Consider the symmetric function*

$$Q_f(x, y) = \frac{1}{2}f(x)^2 + \frac{1}{2}f(y)^2 - f\left(\frac{x+y}{2}\right)^2.$$

*Then the following holds:*

- (1) *if  $(f_k)_{k \in \mathbb{N}}$  is a sequence of real valued convex functions such that  $\sum_{k \in \mathbb{N}} f_k^2$  is convergent then the positive function defined by  $f^2 = \sum_{k \in \mathbb{N}} f_k^2$  is convex and*

$$Q_f = \sum_{k \in \mathbb{N}} Q_{f_k}.$$

*Furthermore given  $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \subseteq X$ , if  $\lim_{n \rightarrow +\infty} Q_f(x_n, y_n) = 0$ , then for every  $k \in \mathbb{N}$*

$$\lim_{n \rightarrow +\infty} Q_{f_k}(x_n, y_n) = 0;$$

- (2) *let  $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \subseteq X$  such that there exists  $C > 0$  with  $|f(x_n)| \leq C$  for every  $n \in \mathbb{N}$ . The following are equivalent:*
- (a)  $\lim_{n \rightarrow +\infty} (f(x_n) - f(y_n)) = 0$  and  $\lim_{n \rightarrow +\infty} (f(\frac{x_n+y_n}{2}) - f(x_n)) = 0$ ;
  - (b)  $\lim_{n \rightarrow +\infty} Q_f(x_n, y_n) = 0$ .

*Proof.* Statement 1. is a straightforward calculation. Assume, without loss of generality, that there exists  $L \in \mathbb{R}$  such that  $\lim_{n \rightarrow +\infty} f(x_n) = L$ . Consider the inequality

$$Q_f(x, y) = \frac{1}{2}f(x)^2 + \frac{1}{2}f(y)^2 - f\left(\frac{x+y}{2}\right)^2 \geq \frac{1}{4}(f(x) - f(y))^2 \geq 0.$$

Then (b) $\Rightarrow$ (a). If (a) holds we have

$$\lim_{n \rightarrow +\infty} f(y_n) = \lim_{n \rightarrow +\infty} f\left(\frac{x_n + y_n}{2}\right) = L,$$

and (b) follows easily. □

The following two lemmata are part of a normalization argument we will use throughout the rest of the paper.

**Lemma 2.2.** *Let  $X$  be a normed space. Consider two norms  $\|\cdot\|_1$  and  $\|\cdot\|_2$  and define  $\|\cdot\|^2 = \|\cdot\|_1^2 + \|\cdot\|_2^2$ . If  $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \subseteq X \setminus \{0\}$  are such that*

$$\lim_{n \rightarrow +\infty} (2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2) = 0 \text{ and } \lim_{n \rightarrow +\infty} \|x_n\|_1 = L$$

for some  $L \neq 0$ , then

$$\lim_{n \rightarrow +\infty} \left( 2 \left\| \frac{x_n}{\|x_n\|_1} \right\|^2 + 2 \left\| \frac{y_n}{\|y_n\|_1} \right\|^2 - \left\| \frac{x_n}{\|x_n\|_1} + \frac{y_n}{\|y_n\|_1} \right\|^2 \right) = 0.$$

*Proof.* By Lemma 2.1 we have

$$(2.1) \quad \lim_{n \rightarrow +\infty} (\|x_n\| - \|y_n\|) = \lim_{n \rightarrow +\infty} (\|x_n + y_n\| - 2\|x_n\|) = 0;$$

$$(2.2) \quad \lim_{n \rightarrow +\infty} (\|x_n\|_1 - \|y_n\|_1) = \lim_{n \rightarrow +\infty} (\|x_n + y_n\|_1 - 2\|x_n\|_1) = 0.$$

By equality (2.1) there exists  $C > 0$  such that  $\max\{\|x_n\|, \|y_n\|\} < C$  for every  $n \in \mathbb{N}$ . By equality (2.1) and equality (2.2), for every  $\varepsilon > 0$  there exists  $n_0 \in \mathbb{N}$  such that

$$(2.3) \quad \begin{aligned} \left| \|x_n\| - \|y_n\| \right| &< \frac{L}{3}\varepsilon, & \left| \|x_n + y_n\| - 2\|x_n\| \right| &< \frac{L}{4}\varepsilon; \\ \left| \frac{1}{L} - \frac{1}{\|y_n\|_1} \right| &< \frac{1}{4C}\varepsilon, & \left| \frac{1}{L} - \frac{1}{\|x_n\|_1} \right| &< \frac{1}{8C}\varepsilon. \end{aligned}$$

By inequalities (2.3), we get for every  $n \geq n_0$

$$\begin{aligned} \left| \left\| \frac{x_n}{\|x_n\|_1} \right\| - \left\| \frac{y_n}{\|y_n\|_1} \right\| \right| &\leq \left| \left\| \frac{x_n}{\|x_n\|_1} \right\| - \left\| \frac{x_n}{L} \right\| \right| + \left| \left\| \frac{x_n}{L} \right\| - \left\| \frac{y_n}{L} \right\| \right| + \left| \left\| \frac{y_n}{L} \right\| - \left\| \frac{y_n}{\|y_n\|_1} \right\| \right| \leq \\ &\leq \left| \frac{1}{\|x_n\|_1} - \frac{1}{L} \right| \|x_n\| + \frac{1}{L} \left| \|x_n\| - \|y_n\| \right| + \left| \frac{1}{L} - \frac{1}{\|y_n\|_1} \right| \|y_n\| \leq \\ &\leq \left| \frac{1}{\|x_n\|_1} - \frac{1}{L} \right| C + \frac{1}{L} \left| \|x_n\| - \|y_n\| \right| + \left| \frac{1}{L} - \frac{1}{\|y_n\|_1} \right| C \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{4} \leq \varepsilon; \end{aligned}$$

and

$$\begin{aligned} \left| \left\| \frac{x_n}{\|x_n\|_1} + \frac{y_n}{\|y_n\|_1} \right\| - 2 \left\| \frac{x_n}{\|x_n\|_1} \right\| \right| &\leq \left| \left\| \frac{x_n}{\|x_n\|_1} + \frac{y_n}{\|y_n\|_1} \right\| - \left\| \frac{x_n}{\|x_n\|_1} + \frac{y_n}{L} \right\| \right| + \\ &+ \left| \left\| \frac{x_n}{\|x_n\|_1} + \frac{y_n}{L} \right\| - \left\| \frac{x_n}{L} + \frac{y_n}{L} \right\| \right| + \left| \left\| \frac{x_n}{L} + \frac{y_n}{L} \right\| - 2 \left\| \frac{x_n}{L} \right\| \right| + 2 \left| \left\| \frac{x_n}{L} \right\| - \left\| \frac{x_n}{\|x_n\|_1} \right\| \right| \leq \\ &\leq \left| \left\| \frac{y_n}{\|y_n\|_1} - \frac{y_n}{L} \right\| \right| + \left| \left\| \frac{x_n}{\|x_n\|_1} - \frac{x_n}{L} \right\| \right| + \left| \left\| \frac{x_n}{L} + \frac{y_n}{L} \right\| - 2 \left\| \frac{x_n}{L} \right\| \right| + 2 \left| \left\| \frac{x_n}{L} \right\| - \left\| \frac{x_n}{\|x_n\|_1} \right\| \right| \leq \\ &\leq \left| \frac{1}{\|y_n\|_1} - \frac{1}{L} \right| \|y_n\| + \left| \frac{1}{\|x_n\|_1} - \frac{1}{L} \right| \|x_n\| + \frac{1}{L} \left| \|x_n + y_n\| - 2\|x_n\| \right| + 2\|x_n\| \left| \frac{1}{L} - \frac{1}{\|x_n\|_1} \right| \leq \\ &\leq \left| \frac{1}{\|y_n\|_1} - \frac{1}{L} \right| C + \left| \frac{1}{\|x_n\|_1} - \frac{1}{L} \right| C + \frac{1}{L} \left| \|x_n + y_n\| - 2\|x_n\| \right| + 2C \left| \frac{1}{L} - \frac{1}{\|x_n\|_1} \right| \leq \varepsilon. \end{aligned}$$

We remark that  $x_n/\|x_n\|_1$  is  $\|\cdot\|$ -bounded. The conclusion follows by Lemma 2.1.  $\square$

**Lemma 2.3.** *Let  $X$  be a normed space,  $F$  a norming subspace of  $X^*$  and  $\|\cdot\|$  an equivalent  $\sigma(X, F)$ -lower semicontinuous norm on  $X$ . Let  $(A, \preceq_A)$  a direct set and  $\{x_\alpha\}_{\alpha \in A} \subseteq X \setminus \{0\}$  a net such that, for some  $y \in X$  and  $L > 0$ ,*

$$\sigma(X, F)\text{-}\lim_{\alpha \in A} x_\alpha = y \text{ and } \lim_{\alpha \in A} \|x_\alpha\| = L.$$

*Then  $\sigma(X, F)\text{-}\lim_{\alpha \in A} x_\alpha/\|x_\alpha\| = y/L$ .*

*Proof.* Let  $f \in F$  and  $\varepsilon > 0$ . Consider  $\alpha_0 \in A$  such that for  $\alpha_0 \preceq_A \alpha$  we have

$$|\|x_\alpha\| - L| < \frac{L}{2\|f\|}\varepsilon \text{ and } |f(x_\alpha - y)| < \frac{L}{2}\varepsilon.$$

We obtain that

$$\begin{aligned} & \left| f\left(\frac{x_\alpha}{\|x_\alpha\|} - \frac{y}{L}\right) \right| \leq \left| f\left(\frac{x_\alpha}{\|x_\alpha\|} - \frac{x_\alpha}{L}\right) \right| + \left| f\left(\frac{x_\alpha}{L} - \frac{y}{L}\right) \right| \leq \\ & \leq \left| \frac{1}{\|x_\alpha\|} - \frac{1}{L} \right| |f(x_\alpha)| + \frac{1}{L} |f(x_\alpha - y)| \leq \frac{1}{L} (|L - \|x_\alpha\||\|f\| + |f(x_\alpha - y)|) < \varepsilon. \end{aligned}$$

□

The following two lemmata are a generalization of [DGZ93, Lemma VII.1.1]. They will be used in the proof of Theorem 1.4.

**Lemma 2.4.** *Let  $X$  be a normed space and  $\theta : X \rightarrow [0, +\infty)$  be a convex, symmetric, uniformly continuous on bounded set with  $\theta(0) = 0$ . An equivalent norm  $\|\cdot\|_\theta$  exists with the following property: if  $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \subseteq X$  satisfy*

$$(2.4) \quad \lim_{n \rightarrow +\infty} (2\|x_n\|_\theta + 2\|y_n\|_\theta - \|x_n + y_n\|_\theta) = 0 \text{ and } (x_n)_{n \in \mathbb{N}} \text{ is bounded.}$$

then

$$\lim_{n \rightarrow +\infty} \left( \frac{1}{2}\theta(x_n) + \frac{1}{2}\theta(y_n) - \theta\left(\frac{x_n + y_n}{2}\right) \right) = 0.$$

Furthermore if  $\theta$  is  $\sigma(X, F)$ -lower semicontinuous, for some norming subspace  $F$ , then  $\|\cdot\|_\theta$  is  $\sigma(X, F)$ -lower semicontinuous.

*Proof.* Consider the absolutely convex sets

$$B_q = \{x \in X \mid \theta(x) \leq q\},$$

which contain the origin as an interior point. For every  $q \in \mathbb{Q}^+$ , let  $\|\cdot\|_q$  the Minkowski functional of the set  $B_q$  and  $c : \mathbb{N} \rightarrow \mathbb{Q}^+$  a one-to-one and onto map. Define

$$\|z\|_\theta^2 = \|z\|^2 + \sum_{n \in \mathbb{N}} \frac{1}{2^n d_{c(n)}^2} \|z\|_{c(n)}^2 \quad \text{for } z \in X$$

where  $d_q$  is a positive number such that  $\|z\|_q \leq d_q \|z\|$ , for every  $q \in \mathbb{Q}^+$ . Consider two sequences  $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \subseteq X$  which satisfy condition (2.4). By Lemma 2.1 for each  $q \in \mathbb{Q}^+$

$$\lim_{n \rightarrow +\infty} (2\|x_n\|_q + 2\|y_n\|_q - \|x_n + y_n\|_q) = 0.$$

For every  $q \in \mathbb{Q}^+$ , by Lemma 2.1, follows

$$(2.5) \quad \lim_{n \rightarrow +\infty} (\|x_n\|_q - \|y_n\|_q) = \lim_{n \rightarrow +\infty} (\|x_n + y_n\|_q - 2\|x_n\|_q) = 0.$$

Fix an increasing sequence of natural numbers  $n_1 < n_2 < \dots < n_k < \dots$ , we are going to prove that there exists a subsequence  $n_{k_1} < n_{k_2} < \dots < n_{k_s} < \dots$  such that

$$\lim_{s \rightarrow +\infty} (\theta(x_{n_{k_s}}) - \theta(y_{n_{k_s}})) = \lim_{s \rightarrow +\infty} \left( \theta\left(\frac{x_{n_{k_s}} + y_{n_{k_s}}}{2}\right) - \theta(x_{n_{k_s}}) \right) = 0.$$



Let  $L$  be a cluster point of  $(\theta(x_{n_k}))_{k \in \mathbb{N}}$ . There exists a subsequence such that

$$(2.6) \quad \lim_{s \rightarrow +\infty} \theta(x_{n_{k_s}}) = L$$

Assume, by contradiction, that  $\lim_{s \rightarrow +\infty} (\theta(x_{n_{k_s}}) - \theta(y_{n_{k_s}})) \neq 0$ , this means that there exists  $\varepsilon_0 > 0$  such that

$$|L - \theta(y_{n_{k_s}})| \geq \varepsilon_0$$

frequently. Assume that there exists a subsequence such that

$$(2.7) \quad \theta(y_{n_{k_{s_l}}}) \leq L - \varepsilon_0.$$

Let  $q_1, q_2 \in \mathbb{Q}^+$  so that  $L - \varepsilon_0 < q_1 < q_2 < L$ . By equality (2.6) and inequality (2.7) follows

$$\|y_{n_{k_{s_l}}}\|_{q_2} \leq 1 \text{ and } \|x_{n_{k_{s_l}}}\|_{q_2} \geq 1$$

eventually.

**Claim.**  $\eta > 0$  exists such that  $\|y_{n_{k_{s_l}}}\|_{q_2} \leq 1 - \eta$  eventually.

Indeed, assume by contradiction that  $\sup_{l \in \mathbb{N}} \|y_{n_{k_{s_l}}}\|_{q_2} = 1$ . Then a subsequence exists such that  $\lim_{m \rightarrow +\infty} \|y_{n_{k_{s_{l_m}}}}\|_{q_2} = 1$ . Since  $\theta$  is  $\|\cdot\|$ -continuous, we deduce that

$$\{x \in X \mid \theta(x) < q_2\} \subseteq \{x \in X \mid \|x\|_{q_2} < 1\}.$$

If set  $y'_m := y_{n_{k_{s_{l_m}}}} / \|y_{n_{k_{s_{l_m}}}}\|_{q_2}$ , then we have  $\theta(y'_m) = q_2$  and  $\lim_{m \rightarrow +\infty} \|y'_m - y_{n_{k_{s_{l_m}}}}\| = 0$ . Since  $\theta$  is uniformly continuous on bounded sets we deduce that

$$\lim_{m \rightarrow +\infty} \theta(y_{n_{k_{s_{l_m}}}}) = q_2,$$

a contradiction with inequality (2.7).

So  $\|y_{n_{k_{s_l}}}\|_{q_2} \leq 1 - \eta$  and  $\|x_{n_{k_{s_l}}}\|_{q_2} \geq 1$  eventually, a contradiction with equalities (2.5). Using a similar argument we obtain the same conclusion, if instead of inequality (2.7) we assume

$$\theta(y_{n_{k_{s_l}}}) \geq L + \varepsilon_0.$$

Repeating the same argument with the sequence  $\frac{1}{2}(x_n + y_n)$ , instead of  $y_n$  we obtain

$$\lim_{s \rightarrow +\infty} (\theta(x_{n_{k_s}}) - \theta(y_{n_{k_s}})) = \lim_{s \rightarrow +\infty} \left( \theta\left(\frac{x_{n_{k_s}} + y_{n_{k_s}}}{2}\right) - \theta(x_{n_{k_s}}) \right) = 0.$$

Since every subsequence of  $(\theta(x_n) - \theta(y_n))_{n \in \mathbb{N}}$  and of  $(\theta(\frac{x_n + y_n}{2}) - \theta(x_n))_{n \in \mathbb{N}}$  have a subsequence converging to zero, we obtain

$$\lim_{n \rightarrow +\infty} (\theta(x_n) - \theta(y_n)) = \lim_{n \rightarrow +\infty} \left( \theta\left(\frac{x_n + y_n}{2}\right) - \theta(x_n) \right) = 0.$$

By Lemma 2.1, the thesis follows.  $\square$

**Lemma 2.5.** *Let  $(\varphi_i)_{i \in I}$ ,  $(\psi_i)_{i \in I}$  be two families of real valued, convex and non-negative functions defined on a normed space  $X$  which are both uniformly bounded on bounded subsets*



of  $X$ . For every  $i \in I$  and  $k \in \mathbb{N}$ , denote

$$\begin{aligned}\theta_{i,k}(x) &= \varphi_i^2(x) + \frac{1}{k}\psi_i^2(x); \\ \theta_k(x) &= \sup_{i \in I} \theta_{i,k}(x); \\ \theta(x) &= \|x\|^2 + \sum_{k \in \mathbb{N}} 2^{-k}(\theta_k(x) + \theta_k(-x)),\end{aligned}$$

where  $\|\cdot\|$  is the norm of  $X$ . An equivalent norm  $\|\cdot\|_\theta$  on  $X$  exists with the following property: if  $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \subseteq X$  satisfy

$$\lim_{n \rightarrow +\infty} (2\|x_n\|_\theta^2 + 2\|y_n\|_\theta^2 - \|x_n + y_n\|_\theta^2) = 0 \text{ and } (x_n)_{n \in \mathbb{N}} \text{ is bounded,}$$

then there exists a sequence  $(i_n)_{n \in \mathbb{N}} \subseteq I$  such that:

- (1)  $\lim_{n \rightarrow +\infty} (\frac{1}{2}\psi_{i_n}^2(x_n) + \frac{1}{2}\psi_{i_n}^2(y_n) - \psi_{i_n}^2(\frac{x_n + y_n}{2})) = 0;$
- (2)  $\lim_{n \rightarrow +\infty} (\varphi_{i_n}(x_n) - \varphi_{i_n}(y_n)) = \lim_{n \rightarrow +\infty} (\varphi_{i_n}(\frac{x_n + y_n}{2}) - \varphi_{i_n}(y_n)) = 0;$
- (3)  $\liminf_{n \rightarrow +\infty} \varphi_{i_n}(x_n) = \liminf_{n \rightarrow +\infty} \sup_{i \in I} \varphi_i(x_n);$
- (4)  $\liminf_{n \rightarrow +\infty} \varphi_{i_n}(y_n) = \liminf_{n \rightarrow +\infty} \sup_{i \in I} \varphi_i(y_n).$

*Proof.* Let  $\|\cdot\|_\theta$  be the norm obtained applying Lemma 2.4. Let  $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \subseteq X$  satisfy

$$(2.8) \quad \lim_{n \rightarrow +\infty} (2\|x_n\|_\theta^2 + 2\|y_n\|_\theta^2 - \|x_n + y_n\|_\theta^2) = 0 \text{ and } (x_n)_{n \in \mathbb{N}} \text{ is bounded.}$$

By Lemma 2.4 we have

$$(2.9) \quad \lim_{n \rightarrow +\infty} \left( \frac{1}{2}\theta(x_n) + \frac{1}{2}\theta(y_n) - \theta\left(\frac{x_n + y_n}{2}\right) \right) = 0.$$

By Lemma 2.1 we have

$$(2.10) \quad \lim_{n \rightarrow +\infty} (2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2) = 0,$$

and for every  $k \in \mathbb{N}$

$$(2.11) \quad \lim_{n \rightarrow +\infty} \left( \frac{1}{2}\theta_k(x_n) + \frac{1}{2}\theta_k(y_n) - \theta_k\left(\frac{x_n + y_n}{2}\right) \right) = 0.$$

Let  $(\alpha_n)_{n \in \mathbb{N}}$  a sequence of real number such that  $\alpha_n > 0$  and  $\lim_{n \rightarrow +\infty} n\alpha_n = 0$ . Apply [DGZ93, Fact VII.1.3] to obtain a sequence  $(k_n)_{n \in \mathbb{N}} \subseteq \mathbb{N}$  and  $N_0 \in \mathbb{N}$  such that

- (i)  $\lim_{n \rightarrow +\infty} k_n = +\infty;$
- (ii)  $\frac{1}{2}\theta_{k_n}(x_n) + \frac{1}{2}\theta_{k_n}(y_n) - \theta_{k_n}(\frac{x_n + y_n}{2}) < \alpha_{k_n}$  for all  $n \geq N_0$ .

It follows from (ii) and the definition of  $\theta_{k_n}$  that for each  $n \in \mathbb{N}$  there exists  $i_n \in I$  such that

$$(2.12) \quad \frac{1}{2}\theta_{k_n}(x_n) + \frac{1}{2}\theta_{k_n}(y_n) - \theta_{i_n, k_n}\left(\frac{x_n + y_n}{2}\right) < \alpha_{k_n}.$$

Thus for all  $i \in I$  we have

$$\begin{aligned}
 \alpha_{k_n} &> \frac{1}{2}\theta_{i,k_n}(x_n) + \frac{1}{2}\theta_{i,k_n}(y_n) - \theta_{i,k_n}\left(\frac{x_n + y_n}{2}\right) \geq \\
 (2.13) \quad &\geq \frac{1}{2}\varphi_{i_n}^2(x_n) + \frac{1}{2}\varphi_{i_n}^2(y_n) - \varphi_{i_n}^2\left(\frac{x_n + y_n}{2}\right) + \frac{1}{2}(\varphi_i^2(x_n) - \varphi_{i_n}^2(x_n)) + \\
 &+ \frac{1}{k_n}\left(\frac{1}{2}\psi_{i_n}^2(x_n) + \frac{1}{2}\psi_{i_n}^2(y_n) - \psi_{i_n}^2\left(\frac{x_n + y_n}{2}\right)\right) + \frac{1}{2k_n}(\psi_i^2(x_n) - \psi_{i_n}^2(x_n));
 \end{aligned}$$

and also

$$\begin{aligned}
 \alpha_{k_n} &> \frac{1}{2}\theta_{i_n,k_n}(x_n) + \frac{1}{2}\theta_{i_n,k_n}(y_n) - \theta_{i_n,k_n}\left(\frac{x_n + y_n}{2}\right) \geq \\
 (2.14) \quad &\geq \frac{1}{2}\varphi_{i_n}^2(x_n) + \frac{1}{2}\varphi_{i_n}^2(y_n) - \varphi_{i_n}^2\left(\frac{x_n + y_n}{2}\right) + \frac{1}{2}(\varphi_i^2(y_n) - \varphi_{i_n}^2(y_n)) + \\
 &+ \frac{1}{k_n}\left(\frac{1}{2}\psi_{i_n}^2(x_n) + \frac{1}{2}\psi_{i_n}^2(y_n) - \psi_{i_n}^2\left(\frac{x_n + y_n}{2}\right)\right) + \frac{1}{2k_n}(\psi_i^2(y_n) - \psi_{i_n}^2(y_n)).
 \end{aligned}$$

If we choose  $i = i_n$  in (2.13) we get

$$(2.15) \quad 0 \leq \left(\frac{\varphi_{i_n}(x_n) - \varphi_{i_n}(y_n)}{2}\right)^2 \leq \frac{1}{2}\varphi_{i_n}^2(x_n) + \frac{1}{2}\varphi_{i_n}^2(y_n) - \varphi_{i_n}^2\left(\frac{x_n + y_n}{2}\right) \leq \alpha_{k_n};$$

$$(2.16) \quad 0 \leq \left(\frac{\psi_{i_n}(x_n) - \psi_{i_n}(y_n)}{2}\right)^2 \leq \frac{1}{2}\psi_{i_n}^2(x_n) + \frac{1}{2}\psi_{i_n}^2(y_n) - \psi_{i_n}^2\left(\frac{x_n + y_n}{2}\right) \leq k_n\alpha_{k_n}.$$

Since  $\lim_{n \rightarrow +\infty} k_n\alpha_{k_n} = 0$ , inequality (2.16) implies 1. Furthermore Lemma 2.1 and inequality (2.15) imply 2, i.e.

$$\lim_{n \rightarrow +\infty} (\varphi_{i_n}(x_n) - \varphi_{i_n}(y_n)) = \lim_{n \rightarrow +\infty} \left(\varphi_{i_n}\left(\frac{x_n + y_n}{2}\right) - \varphi_{i_n}(y_n)\right) = 0.$$

Let  $M_n := \sup_{i \in I} \psi_i^2(y_n)$  and  $M := \sup_{n \in \mathbb{N}} M_n$ . For every  $n \in \mathbb{N}$  and  $i \in I$  inequality (2.14) tells us

$$\varphi_i^2(y_n) - \varphi_{i_n}^2(y_n) \leq 2\alpha_{k_n} + \frac{\psi_{i_n}^2(y_n)}{k_n} - \frac{\psi_i^2(y_n)}{k_n} \leq 2\alpha_{k_n} + \frac{M}{k_n}.$$

Thus, for every  $n \in \mathbb{N}$  we have

$$\varphi_{i_n}^2(y_n) \geq \sup_{i \in I} \varphi_i^2(y_n) - \frac{M}{k_n} - 2\alpha_{k_n}.$$

Hence  $\liminf_{n \rightarrow +\infty} \varphi_{i_n}(y_n) = \liminf_{n \rightarrow +\infty} \sup_{i \in I} \varphi_i(y_n)$ . Using the same argument, by inequality (2.13) we have

$$\liminf_{n \rightarrow +\infty} \varphi_{i_n}(x_n) = \liminf_{n \rightarrow +\infty} \sup_{i \in I} \varphi_i(x_n).$$

□

Now we have all the ingredients necessary to prove Theorem 1.4.

*Proof of Theorem 1.4.* For  $H \in \mathcal{H}$  let

$$\varphi_H(x) := \inf \left\{ \|x - c\|_F \mid c \in \overline{\text{conv}(A) \cap (X \setminus H)}^{\sigma(X^{**}, X^*)} \right\} \quad \text{for } x \in X.$$

By [OT09a, Proposition 2.1],  $\varphi_H$  are convex and  $\sigma(X, F)$ -lower semicontinuous functions. For a fixed  $a_H \in A \cap H$  let

$$D_H := \text{conv}(A \cap H) \text{ and } D_{H,\delta} := D_H + \delta \mathcal{B}_X,$$

for every  $H \in \mathcal{H}$  and  $\delta > 0$ . We are going to denote with  $p_{H,\delta}$  the Minkowski functional of the convex body  $\overline{D_{H,\delta}}^{\sigma(X,F)} - a_H$ . Let

$$p_H^2(x) = \|x\|^2 + \sum_{n \in \mathbb{N}} \frac{1}{n^2 2^n} p_{H,1/n}^2(x) \quad \text{for } x \in X.$$

Since  $\delta \mathcal{B}_X \subseteq \overline{D_{H,\delta}}^{\sigma(X,F)} - a_H$  for every  $H \in \mathcal{H}$  and  $\delta > 0$ , then

$$p_{H,\delta} \left( \delta \frac{x}{\|x\|_F} \right) \leq 1 \implies p_{H,\delta}(x) \leq \frac{\|x\|_F}{\delta},$$

and the series converges. Finally consider the convex and  $\sigma(X, F)$ -lower semicontinuous functions

$$\psi_H(x) := p_H(x - a_H).$$

Apply [DGZ93, Lemma VII.1.1] and Lemma 2.5 both with the families  $(\varphi_H)_{H \in \mathcal{H}}$  and  $(\psi_H)_{H \in \mathcal{H}}$  to get two equivalent  $\sigma(X, F)$ -lower semicontinuous norms  $\|\cdot\|_L$  and  $\|\cdot\|_U$ , respectively. Consider the equivalent  $\sigma(X, F)$ -lower semicontinuous norm

$$\|x\|_{A,\mathcal{H}}^2 = \|x\|_L^2 + \|x\|_U^2 \quad \text{for } x \in X.$$

Assume that  $x \in X$  and  $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \subseteq X$  satisfies

$$\begin{aligned} \lim_{n \rightarrow +\infty} (2\|x\|_{A,\mathcal{H}}^2 + 2\|x_n\|_{A,\mathcal{H}}^2 - \|x + x_n\|_{A,\mathcal{H}}^2) &= 0; \\ \lim_{n \rightarrow +\infty} (2\|x_n\|_{A,\mathcal{H}}^2 + 2\|y_n\|_{A,\mathcal{H}}^2 - \|x_n + y_n\|_{A,\mathcal{H}}^2) &= 0. \end{aligned}$$

By Lemma 2.1 we obtain

$$\begin{aligned} \lim_{n \rightarrow +\infty} (2\|x\|_L^2 + 2\|x_n\|_L^2 - \|x + x_n\|_L^2) &= 0; \\ \lim_{n \rightarrow +\infty} (2\|x_n\|_U^2 + 2\|y_n\|_U^2 - \|x_n + y_n\|_U^2) &= 0, \end{aligned}$$

so the thesis of [DGZ93, Lemma VII.1.1] and Lemma 2.5 assure the existence of two sequences  $(H_n^L)_{n \in \mathbb{N}}, (H_n^U)_{n \in \mathbb{N}} \subseteq \mathcal{H}$  such that

- (i)  $\lim_{n \rightarrow +\infty} \left( \frac{1}{2} \psi_{H_n^L}^2(x) + \frac{1}{2} \psi_{H_n^L}^2(x_n) - \psi_{H_n^L}^2\left(\frac{x+x_n}{2}\right) \right) = 0;$
- (ii)  $\lim_{n \rightarrow +\infty} \left( \frac{1}{2} \psi_{H_n^U}^2(x_n) + \frac{1}{2} \psi_{H_n^U}^2(y_n) - \psi_{H_n^U}^2\left(\frac{x_n+y_n}{2}\right) \right) = 0;$
- (iii)  $\lim_{n \rightarrow +\infty} \varphi_{H_n^L}(x) = \lim_{n \rightarrow +\infty} \varphi_{H_n^L}(x_n) = \lim_{n \rightarrow +\infty} \varphi_{H_n^L}\left(\frac{x+x_n}{2}\right) = \sup_{H \in \mathcal{H}} \varphi_H(x);$
- (iv)  $\lim_{n \rightarrow +\infty} (\varphi_{H_n^U}(x_n) - \varphi_{H_n^U}(y_n)) = \lim_{n \rightarrow +\infty} (\varphi_{H_n^U}\left(\frac{x_n+y_n}{2}\right) - \varphi_{H_n^U}(y_n));$
- (v)  $\liminf_{n \rightarrow +\infty} \varphi_{H_n^U}(x_n) = \liminf_{n \rightarrow +\infty} \sup_{H \in \mathcal{H}} \varphi_H(x_n);$
- (vi)  $\liminf_{n \rightarrow +\infty} \varphi_{H_n^U}(y_n) = \liminf_{n \rightarrow +\infty} \sup_{H \in \mathcal{H}} \varphi_H(y_n).$

We need now to prove two claims:

**Claim 1.** There exists  $\eta > 0$  such that

$$\sup_{H \in \mathcal{H}} \varphi_H(x) > \eta, \quad \liminf_{n \rightarrow +\infty} \varphi_{H_n^U}(x_n) \geq \eta, \quad \text{and} \quad \liminf_{n \rightarrow +\infty} \varphi_{H_n^U}(y_n) \geq \frac{\eta}{4}.$$

*Proof.* We know that  $H_0 \in \mathcal{H}$  exists such that  $x \in A \cap H_0$ . Thus  $\varphi_{H_0}(x) > 0$ , so  $\eta > 0$  exists such that

$$\sup_{H \in \mathcal{H}} \varphi_H(x) \geq \varphi_{H_0}(x) > \eta > 0.$$

Condition (iii) implies that there exists  $n_1 \in \mathbb{N}$  such that

$$(2.17) \quad \varphi_{H_n^L}(x) > \eta \quad \text{and} \quad \varphi_{H_n^L}(x_n) > \eta \quad \text{for } n \geq n_1,$$

in particular for every  $n \geq n_1$  follows  $\sup_{H \in \mathcal{H}} \varphi_H(x_n) \geq \eta$ . So we have that

$$(2.18) \quad \liminf_{n \rightarrow +\infty} \sup_{H \in \mathcal{H}} \varphi_H(x_n) \geq \eta$$

Condition (v) and inequality (2.18) implies that eventually  $\varphi_{H_n^U}(x_n) > \eta/2$ . By (iv) follows that eventually

$$|\varphi_{H_n^U}(x_n) - \varphi_{H_n^U}(y_n)| < \frac{\eta}{4},$$

in particular eventually

$$\varphi_{H_n^U}(y_n) > \varphi_{H_n^U}(x_n) - \frac{\eta}{4} > \frac{\eta}{4}.$$

Thus  $n_2 \in \mathbb{N}$  exists such that

$$\sup_{H \in \mathcal{H}} \varphi_H(y_n) \geq \frac{\eta}{4} \quad \text{for } n \geq n_2,$$

and so by (vi) and a classical lower limit property we obtain

$$\liminf_{n \rightarrow +\infty} \varphi_{H_n^U}(y_n) = \liminf_{n \rightarrow +\infty} \sup_{H \in \mathcal{H}} \varphi_H(y_n) \geq \frac{\eta}{4}.$$

□

**Claim 2.** If  $y \in X$  and  $H \in \mathcal{H}$  are such that  $y \in A \cap H$ , then for every  $\delta > 0$

$$p_{H,\delta}(y - a_H) < \frac{\|y - a_H\|}{\delta + \|y - a_H\|}.$$

*Proof.* Obviously we have

$$y - a_H + \delta \frac{y - a_H}{\|y - a_H\|} \in D_{H,\delta} - a_H.$$

So we obtain

$$p_{H,\delta}\left(y - a_H + \delta \frac{y - a_H}{\|y - a_H\|}\right) < 1 \implies p_{H,\delta}(y - a_H) < \frac{\|y - a_H\|}{\delta + \|y - a_H\|}.$$

□

Now we can prove our thesis:

- (1) It follows by inequalities (2.17).

(2) For every  $q \in \mathbb{N}$ , condition (i) and Lemma 2.1 implies

$$\lim_{n \rightarrow +\infty} \left( \frac{1}{2} p_{H_n^L, 1/q}^2(x - a_{H_n^L}) + \frac{1}{2} p_{H_n^L, 1/q}^2(x_n - a_{H_n^L}) - p_{H_n^L, 1/q}^2\left(\frac{x + x_n}{2} - a_{H_n^L}\right) \right) = 0.$$

By Lemma 2.1 we have

$$(2.19) \quad \lim_{n \rightarrow +\infty} (p_{H_n^L, 1/q}(x - a_{H_n^L}) - p_{H_n^L, 1/q}(x_n - a_{H_n^L})) = 0.$$

Let us choose an integer  $q \in \mathbb{N}$  such that  $1/q < \delta$  and  $n \geq n_L$ . By Claim 2, since  $x \in A \cap H_n^L$  for every  $n \geq n_L$ , follows

$$p_{H_n^L, 1/q}(x - a_{H_n^L}) < \frac{\|x - a_{H_n^L}\|}{\frac{1}{q} + \|x - a_{H_n^L}\|} \quad \text{for } n \geq n_L.$$

By the boundedness of  $A$  we can find a number  $0 < \xi \leq 1$  such that

$$p_{H_n^L, 1/q}(x - a_{H_n^L}) < 1 - \xi \quad \text{for } n \geq n_L.$$

By equality (2.19) there exists  $n_{L,\delta} \in \mathbb{N}$  such that

$$p_{H_n^L, 1/q}(x_n - a_{H_n^L}) < 1 - \xi \quad \text{for } n \geq n_{L,\delta}.$$

Thus we arrive to the fact that  $x_n - a_{H_n^L} \in D_{H_n^L, \delta} - a_{H_n^L}$  and indeed

$$x_n \in \text{conv}(A \cap H_n^L) + \delta \mathcal{B}_X \quad \text{for } n \geq n_{L,\delta}.$$

(3) By Claim 1 there exists  $k_U \in \mathbb{N}$  such that

$$\varphi_{H_{n_k}^U}(x_{n_k}) > \frac{\eta}{8} \quad \text{and} \quad \varphi_{H_{n_k}^U}(y_{n_k}) > \frac{\eta}{8}$$

for  $k \geq k_U$ . Without loss of generality we can assume that  $k_U \geq k_0$ . Since for  $k \geq k_U$  we know that  $x_{n_k}, y_{n_k} \in A$ , we have

$$x_{n_k}, y_{n_k} \in H_{n_k}^U \quad \text{for } k \geq k_U.$$

(4) Without loss of generality we can assume that  $x_{n_k} \in A \cap \bigcup \mathcal{H}$  for  $k \geq k_0$ . By conclusion (3) we have

$$x_{n_k} \in H_{n_k}^U \quad \text{for } k \geq k_U.$$

For every  $q \in \mathbb{N}$ , condition (ii) and Lemma 2.1 imply

$$(2.20) \quad \lim_{k \rightarrow +\infty} (p_{H_{n_k}^U, 1/q}(x_{n_k} - a_{H_{n_k}^U}) - p_{H_{n_k}^U, 1/q}(y_{n_k} - a_{H_{n_k}^U})) = 0.$$

Fix  $q \in \mathbb{N}$  such that  $1/q < \delta$ , by Claim 2 we have

$$p_{H_{n_k}^U, 1/q}(x_{n_k} - a_{H_{n_k}^U}) < \frac{\|x_{n_k} - a_{H_{n_k}^U}\|}{\frac{1}{q} + \|x_{n_k} - a_{H_{n_k}^U}\|}.$$

By the boundness of  $A$  we can find  $\xi \in (0, 1]$  with

$$(2.21) \quad p_{H_{n_k}^U, 1/q}(x_{n_k} - a_{H_{n_k}^U}) < 1 - \xi \quad \text{for } k \geq k_U.$$

By equality (2.20) and inequality (2.21) there exists  $k_{\varepsilon, \delta} \in \mathbb{N}$  such that

$$p_{H_{n_k}^U, 1/q}(y_{n_k} - a_{H_{n_k}^U}) < 1 - \xi \quad \text{for } k \geq k_{U, \delta}.$$

For every  $k \geq k_{U,\delta}$  we have  $y_{n_k} - a_{H_{n_k}^U} \in D_{H_{n_k}^U, \delta} - a_{H_{n_k}^U}$ . So

$$y_{n_k} \in \text{conv}(A \cap H_{n_k}^U) + \delta \mathcal{B}_X \quad \text{for } k \geq k_{U,\delta}.$$

□

### 3. A SLICE DERIVATION PROCESS

We want to define a derivation process in the spirit of the ones used by G. Lancien and M. Raja to construct equivalent norms (see for example [Lan93], [Lan95], [Raj07] and [Raj13]). Let  $X$  be a normed space and  $F \subseteq X^*$  a norming subspace. Fix a family  $\Lambda$  of  $\sigma(X, F)$ -open sets of  $\mathcal{B}_X$ ,  $\mathcal{H}_F$  the family of all  $\sigma(X, F)$ -open half-spaces of  $\mathcal{B}_X$  and for a set  $T \subseteq \mathcal{B}_X$  define

$$\text{Sl}_F(T, \Lambda) := \{H \cap T \mid H \in \mathcal{H}_F \text{ and there exists } W \in \Lambda \text{ with } \emptyset \neq H \cap T \subseteq W\}.$$

Set  $B_F^{(0)}(\mathcal{B}_X, \Lambda) := \mathcal{B}_X$  and by transfinite induction for every ordinal  $\alpha$  we let

$$B_F^{(\alpha+1)}(\mathcal{B}_X, \Lambda) := B_F^{(\alpha)}(\mathcal{B}_X, \Lambda) \setminus \bigcup \text{Sl}_F(B_F^{(\alpha)}(\mathcal{B}_X, \Lambda), \Lambda),$$

whenever  $B_F^{(\alpha)}(\mathcal{B}_X, \Lambda) \neq \emptyset$ . If  $\lambda$  is a limit ordinal then

$$B_F^{(\lambda)}(\mathcal{B}_X, \Lambda) := \bigcap_{\alpha < \lambda} B_F^{(\alpha)}(\mathcal{B}_X, \Lambda).$$

**Lemma 3.1** (Eating lemma). *Let  $X^*$  a dual Banach space and  $\Lambda$  a family of  $w^*$ -open sets. It follows  $B_X^{(\omega_0)}(\mathcal{B}_{X^*}, \Lambda) = \overline{\text{conv}}^{w^*}(\mathcal{B}_{X^*} \setminus \bigcup \Lambda)$ , where  $\omega_0$  is the first infinite ordinal.*

*Proof.* Throughout the proof we set  $B^{(\alpha)} := B_X^{(\alpha)}(\mathcal{B}_{X^*}, \Lambda)$  for every ordinal  $\alpha$ . It is easy to see that  $\mathcal{B}_{X^*} \setminus \bigcup \Lambda \subseteq B^{(\omega_0)}$ . By the Krein–Milman theorem the thesis follows if we prove that  $\text{ext}(B^{(\omega_0)}) \subseteq \mathcal{B}_{X^*} \setminus \bigcup \Lambda$ . Assume, by contradiction, that there exists  $x_0^* \in \text{ext}(B^{(\omega_0)}) \cap \bigcup \Lambda$ , then we have

$$(3.1) \quad x_0^* \notin \bigcup \text{Sl}_X(B^{(n)}, \Lambda) \quad \text{for every } n \in \mathbb{N}.$$

Fix  $W \in \Lambda$  such that  $x_0^* \in W$ . By Choquet's lemma (see [FHH<sup>+</sup>11, Lemma 3.69]), there exists a  $w^*$ -open half-space  $H$  such that

$$(3.2) \quad x_0^* \in H \cap B^{(\omega_0)} \subseteq W.$$

Let  $H_1$  be a  $w^*$ -open half-space such that  $x_0^* \in H_1 \subseteq \overline{H_1}^{w^*} \subseteq H$ , by (3.1) there exists  $x_n^* \in H_1 \cap B^{(n)}$  such that  $x_n^* \notin W$  for every  $n \in \mathbb{N}$ . By the  $w^*$ -compactness of the dual unit ball there is a cluster point  $x^*$  of the sequence  $(x_n^*)_{n \in \mathbb{N}}$ , thus

$$x^* \in B^{(n)} \cap \overline{H_1}^{w^*} \text{ and } x^* \notin W.$$

So  $x^* \in B^{(\omega_0)} \cap \overline{H_1}^{w^*} \subseteq B^{(\omega_0)} \cap H$ , which is a contradiction with (3.2). □

The previous lemma says that if  $\Lambda$  is a  $w^*$ -open cover of  $\mathcal{B}_{X^*}$ , then  $B^{(\omega_0)}(\mathcal{B}_{X^*}, \Lambda) = \emptyset$ . Now we want to prove that  $B^{(\omega_0)}(\mathcal{B}_{X^*}, \Lambda) \cap \mathcal{S}_{X^*} = \emptyset$ , whenever  $\Lambda$  is an open cover of the unit sphere. To this goal we need the following extreme point lemma of Choquet (see [Cho69, Lemma 27.8]). We state it here for the sake of completeness.

**Lemma 3.2** (Choquet's extreme point lemma). *Let  $X$  a Hausdorff topological vector space,  $C \subseteq X$  a convex set and  $A \subseteq C$  a convex and linearly compact set (that is, any line intersecting  $A$  does so in a closed segment). If  $B = C \setminus A$  is convex and  $\text{ext}(A) \neq \emptyset$ , then we have  $\text{ext}(A) \cap \text{ext}(C) \neq \emptyset$ .*

Now we can prove that our derivation process “eats” the whole unit sphere in at most  $\omega_0$  steps.

**Lemma 3.3** (Eating lemma for the sphere). *Let  $X^*$  be a dual Banach space. If  $\Lambda$  is a  $w^*$ -open cover of the unit sphere, then  $B_X^{(\omega_0)}(\mathcal{B}_{X^*}, \Lambda) \cap \mathcal{S}_{X^*} = \emptyset$ .*

*Proof.* Throughout the proof we set  $B^{(\alpha)} := B_X^{(\alpha)}(\mathcal{B}_{X^*}, \Lambda)$  for every ordinal  $\alpha$ . Assume, by contradiction, that there exists  $x^* \in B^{(\omega_0)} \cap \mathcal{S}_{X^*}$ . Let  $n \in \mathbb{N}$  and consider

$$B_n^{(\omega_0)} = B^{(\omega_0)} \cap \left(1 - \frac{1}{n+1}\right) \mathcal{B}_{X^*}.$$

Without loss of generality we can, and do, assume that  $B_1^{(\omega_0)} \neq \emptyset$ . We plan to construct a family of slices which press the point  $x^*$  in such a way that an extreme point  $x_0^* \in \text{ext}(B^{(\omega_0)}) \cap \mathcal{S}_{X^*}$  can be found. Then we can conclude our proof as in Lemma 3.1. We start with a preliminary construction which enable us to apply Lemma 3.2. By the Hahn–Banach theorem, for every  $n \in \mathbb{N}$  a  $w^*$ -open halfspace  $H_n$  exists with the following properties:

- (1)  $x^* \in H_n$ ,
- (2)  $B_1^{(\omega_0)} \subseteq B^{(\omega_0)} \setminus \overline{H_1}^{w^*}$ ,
- (3)  $\text{conv}((B^{(\omega_0)} \setminus H_n) \cup B_n^{(\omega_0)}) \subseteq B^{(\omega_0)} \setminus \overline{H_{n+1}}^{w^*}$  for every  $n \in \mathbb{N}$ .

This is possible due to the fact that  $x^* \notin \text{conv}((B^{(\omega_0)} \setminus H_n) \cup B_n^{(\omega_0)})$  and every one of the sets  $\text{conv}((B^{(\omega_0)} \setminus H_n) \cup B_n^{(\omega_0)})$  is  $w^*$ -closed. Let us consider the convex and  $w^*$ -compact set

$$H = \bigcap_{n \in \mathbb{N}} (\overline{H_n}^{w^*} \cap B^{(\omega_0)}).$$

It is easy to see that  $H \subseteq \mathcal{S}_{X^*}$  and  $B^{(\omega_0)} \setminus H$  is convex, then by the Krein–Milman theorem and Lemma 3.2 we obtain that there exists  $x_0^* \in \text{ext}(H) \cap \text{ext}(B^{(\omega_0)})$ . Now following the proof of Lemma 3.1, we arrive to a contradiction which finishes the proof.  $\square$

#### 4. SPHERES HAVING A $G_\delta$ -DIAGONAL

The topological condition we are going to relate with the existence of an equivalent rotund norm with some lower semicontinuity conditions is the following (see [Gru84, Section 2]).

**Definition 4.1.** Let  $(X, \tau)$  be a topological space. We say that  $X$  has a  $G_\delta$ -diagonal for  $\tau$  if, and only if, the set  $\Delta = \{(x, x) \mid x \in X\}$  is a  $G_\delta$ -set in  $X \times X$ , with the product topology.

The following well known theorem (see [Ced61, Lemma 5.4] or [Gru84, Theorem 2.2]) will be usefull for our purposes.

**Theorem 4.2.** *Let  $(X, \tau)$  be a topological space.  $X$  has a  $G_\delta$ -diagonal for  $\tau$  if, and only if, there exists a sequence  $(\mathcal{G}_n)_{n \in \mathbb{N}}$  of open covers of  $X$  such that for each  $x, y \in X$  with  $x \neq y$ ,*



there exists  $n \in \mathbb{N}$  with

$$y \notin \text{st}(x, \mathcal{G}_n) := \bigcup \{U \in \mathcal{G}_n \mid x \in U\}.$$

We will call the sequence  $(\mathcal{G}_n)_{n \in \mathbb{N}}$  a  $G_\delta$ -diagonal sequence.

The next proposition is borrowed from [ST10, Proposition 5]. We prove it just for the sake of completeness.

**Proposition 4.3.** *Let  $X$  a normed space and  $F \subseteq X^*$  a norming subspace. If  $X$  admits an equivalent,  $\sigma(X, F)$ -lower semicontinuous and rotund norm  $\|\cdot\|_R$ , then its unit sphere  $S_R$  has a  $G_\delta$ -diagonal for the  $\sigma(X, F)$ -topology. Furthermore a  $G_\delta$ -diagonal sequence  $(\mathcal{G}_n)_{n \in \mathbb{N}}$  can be obtained such that for every  $n \in \mathbb{N}$  the members of  $\mathcal{G}_n$  are open slices of  $S_R$ .*

*Proof.* Let  $S_R^*$  be the dual sphere related with the norm  $\|\cdot\|_R$ . Given a rational  $q \in \mathbb{Q} \cap (0, 1)$  consider the families of  $\sigma(X, F)$ -open slices

$$\mathcal{H}_q = \{\{x \in S_R \mid f(x) > q\} \mid f \in S_R^* \cap F\}.$$

Consider two distinct point  $x, y \in S_R$  and  $q_0 \in \mathbb{Q} \cap (0, 1)$  such that  $\|\frac{1}{2}(x + y)\|_R < q_0 < 1$ , it is obvious that every  $H \in \mathcal{H}_{q_0}$  cannot contain both  $x$  and  $y$ .  $\square$

When a situation like this happens we will say that  $S_R$  has a  $G_\delta$ -diagonal for the  $\sigma(X, F)$ -topology, with  $\sigma(X, F)$ -slices. We are now able to prove Theorem 1.5.

*Proof of Theorem 1.5.* Proposition 4.3 gives us (1)  $\Rightarrow$  (2) and (1)  $\Rightarrow$  (3).

Assume (2) holds and let  $S_\delta$  be the unit sphere of the norm  $\|\cdot\|_\delta$ . Let  $(\mathcal{H}_n)_{n \in \mathbb{N}}$  be a  $G_\delta$ -diagonal sequence for  $S_\delta$ , made of  $\sigma(X, F)$ -slices. Apply Theorem 1.3 with  $S_\delta$  and  $\mathcal{H}_n$  in order to obtain a countable number of equivalent norms  $\|\cdot\|_n$  which satisfy the conclusion of Theorem 1.3. Consider the  $\sigma(X, F)$ -lower semicontinuous norm

$$\|\cdot\|_R^2 = \|\cdot\|_\delta^2 + \sum_{n \in \mathbb{N}} c_n \|\cdot\|_n^2,$$

where  $c_n$  are positive constants taken in such a way that the series converges uniformly on bounded subsets of  $X$ . Indeed, there are constants  $a_n, b_n$  such that  $a_n \|\cdot\| \leq \|\cdot\|_n \leq b_n \|\cdot\|$ , for every  $n \in \mathbb{N}$ , it suffices to take  $c_n = 1/(2^n b_n)$ . Let  $x, y \in X$  and assume that  $2\|x\|_R^2 + 2\|y\|_R^2 - \|x + y\|_R^2 = 0$ . By contradiction, assume that  $x \neq y$ . By Lemma 2.1 we obtain, for every  $n \in \mathbb{N}$

$$(\dagger_n) \quad 2\|x\|_n^2 + 2\|y\|_n^2 - \|x + y\|_n^2 = 0,$$

as well as  $\|x\|_\delta = \|y\|_\delta$ . Dividing every equation  $(\dagger_n)$  by  $\|x\|_\delta^2 = \|y\|_\delta^2$  we obtain

$$(\dagger_n^\delta) \quad 2\left\|\frac{x}{\|x\|_\delta}\right\|_n^2 + 2\left\|\frac{y}{\|y\|_\delta}\right\|_n^2 - \left\|\frac{x}{\|x\|_\delta} + \frac{y}{\|y\|_\delta}\right\|_n^2 = 0.$$

We know that there exists  $n_0$  such that  $y/\|y\|_\delta \notin \text{st}(x/\|x\|_\delta, \mathcal{H}_{n_0})$ . By Theorem 1.3 and  $(\dagger_{n_0}^\delta)$ , there exists  $H \in \mathcal{H}_{n_0}$  such that  $x/\|x\|_\delta, y/\|y\|_\delta \in H \cap S_\delta$ . A contradiction, so  $x = y$  and  $\|\cdot\|_R$  is  $\sigma(X, F)$ -lower semicontinuous and rotund.

In the dual case Lemma 3.3 allows us to replace open slices by open neighbourhoods. Assume (3) holds. Throughout the proof we will use the notation of Section 3. Let  $(\mathcal{U}_n)_{n \in \mathbb{N}}$

a  $G_\delta$ -diagonal sequence for  $S_G$ . Let  $B_G := \{x^* \in X^* \mid \|x^*\|_G \leq 1\}$  and  $B_n^{(\alpha)} := B_X^{(\alpha)}(B_G, \mathcal{U}_n)$  for every ordinal  $\alpha$ . By Lemma 3.3 we know that

$$B_n^{(\omega_0)} \cap S_G = \emptyset.$$

For every  $n, m \in \mathbb{N}$  apply Theorem 1.3 to the set  $B_n^{(m)}$  and the family  $\text{Sl}_X(B_n^{(m)}, \mathcal{U}_n)$  and let  $\|\cdot\|_{n,m}$  the norm obtained. Consider the dual norm

$$\|z\|_R^2 = \|z\|_G^2 + \sum_{n,m \in \mathbb{N}} c_{n,m} \|z\|_{n,m}^2 \quad \text{for } z \in X$$

where  $c_{n,m}$  are positive constants taken in such a way that the series converges uniformly on bounded subsets of  $X$ . Let's now prove rotundness. Let  $x^*, y^* \in X^*$  such that

$$(4.1) \quad 2\|x^*\|_R^2 + 2\|y^*\|_R^2 - \|x^* + y^*\|_R^2 = 0.$$

If  $x^* = 0$ , then condition (4.1) implies  $y^* = 0$ . Assume  $x^* \neq 0$ , by Lemma 2.1 and condition (4.1) we have for every  $n, m \in \mathbb{N}$

$$(4.2) \quad 2\|x^*\|_G^2 + 2\|y^*\|_G^2 - \|x^* + y^*\|_G^2 = 0;$$

$$(4.3) \quad 2\|x^*\|_{n,m}^2 + 2\|y^*\|_{n,m}^2 - \|x^* + y^*\|_{n,m}^2 = 0.$$

By condition (4.2), we have  $\|x^*\|_G = \|y^*\|_G$ . For every  $n, m \in \mathbb{N}$  dividing equation (4.3) by  $\|x^*\|_G^2 = \|y^*\|_G^2$  we have

$$(\dagger_{n,m}) \quad 2\left\|\frac{x^*}{\|x^*\|_G}\right\|_{n,m}^2 + 2\left\|\frac{y^*}{\|y^*\|_G}\right\|_{n,m}^2 - \left\|\frac{x^*}{\|x^*\|_G} + \frac{y^*}{\|y^*\|_G}\right\|_{n,m}^2 = 0.$$

For  $z \in S_G$  let

$$m_n(z) := \min\{m \in \mathbb{N} \mid z \in B_n^{(m)} \text{ and } z \notin B_n^{(m+1)}\},$$

by Lemma 3.3  $m_n(z)$  exists and is finite for every  $n \in \mathbb{N}$  and  $z \in S_G$ . Assume, by contradiction, that  $x^* \neq y^*$  and let  $n_0 \in \mathbb{N}$  such that

$$(4.4) \quad \frac{y^*}{\|y^*\|_G} \notin \text{st}\left(\frac{x^*}{\|x^*\|_G}, \mathcal{U}_{n_0}\right)$$

Without loss of generality we can assume that

$$(4.5) \quad m_{n_0}\left(\frac{x^*}{\|x^*\|_G}\right) \geq m_{n_0}\left(\frac{y^*}{\|y^*\|_G}\right) := m_0.$$

Condition (4.5) implies

$$(4.6) \quad \frac{x^*}{\|x^*\|_G} \in B_{n_0}^{(m_0)} \quad \text{and} \quad \frac{y^*}{\|y^*\|_G} \in B_{n_0}^{(m_0)} \cap \bigcup \text{Sl}_X(B_{n_0}^{(m_0)}, \mathcal{U}_{n_0}).$$

By the thesis of Theorem 1.3, condition  $(\dagger_{n_0, m_0})$  and condition (4.6) there exists  $H \cap B_{n_0}^{(m_0)} \in \text{Sl}_X(B_{n_0}^{(m_0)}, \mathcal{U}_{n_0})$  such that

$$\frac{x^*}{\|x^*\|_G}, \frac{y^*}{\|y^*\|_G} \in H \cap B_{n_0}^{(m_0)}.$$

By definition  $W \in \mathcal{U}_{n_0}$  exists such that  $H \cap B_{n_0}^{(m_0)} \subseteq W$ . A contradiction with (4.4). So  $x^* = y^*$  and  $\|\cdot\|_R$  is an equivalent dual rotund norm.  $\square$

The previous result appears as an improvement of [MOTZ07, Theorem 1.2]. Remember that a norm is said to be  $\sigma(X, F)$ -Kadec if the norm and the  $\sigma(X, F)$ -topologies coincides when restricted on the unit sphere (see [Raj99b] and [FOOR16]). Theorem 1.5 allows us to prove the following fact, which was already proved in [Raj02, Theorem 1.3] in a different way.

**Corollary 4.4.** *Let  $X^*$  a dual Banach space.  $X^*$  admits an equivalent  $w^*$ -Kadec norm if, and only if,  $X^*$  admits an equivalent dual LUR norm.*

*Proof.* It is a known fact that a dual LUR norm is  $w^*$ -Kadec (see [DGZ93, Proposition II.1.4]). Let  $\|\cdot\|_*$  be an equivalent  $w^*$ -Kadec norm on  $X^*$ . Recall that  $\|\cdot\|_*$  is a dual norm (see [Raj99a, Proposition 4]) and the  $w^*$ -topology is metrizable when restricted to its unit sphere. A metrizable space has a  $G_\delta$ -diagonal, so the existence of an equivalent dual rotund norm follows by Theorem 1.5. The thesis follows by [Raj99a, Theorem 2].  $\square$

## 5. SYMMETRICS ON THE UNIT SPHERE

One of the most famous result in LUR renorming theory is the following theorem by S. Troyanski (see [Tro79] and [Raj99b]).

**Theorem 5.1.** *Let  $X$  a Banach space and  $F$  a norming subspace in  $X^*$ .  $X$  admits an equivalent  $\sigma(X, F)$ -lower semicontinuous LUR norm if, and only if, there exists and equivalent  $\sigma(X, F)$ -lower semicontinuous norm  $\|\cdot\|_D$  such that every point of its unit sphere  $S_D$  is  $F$ -denting, i.e. for every  $\varepsilon > 0$  and  $x \in S_D$  there exists a  $\sigma(X, F)$ -open halfspace  $H$  such that  $x \in H$  and*

$$\|\cdot\|_D\text{-diam}(H \cap B_D) < \varepsilon,$$

where  $B_D$  is the unit ball of  $\|\cdot\|_D$ .

In what follows we will generalize the concept of denting point, but in order to do so we need a classical topological concept (see [Gru84, Section 9]).

**Definition 5.2.** Let  $S$  be a nonempty set. A function  $\rho : S \times S \rightarrow [0, +\infty)$  is called *symmetric* if  $\rho(x, y) = \rho(y, x)$  for every  $x, y \in S$ , and  $\rho(x, y) = 0$  if, and only if,  $x = y$ .

If a set  $S$  has a symmetric  $\rho$ , then we can define a topology  $\tau_\rho$  in the following way:  $U$  is an open set in the  $\tau_\rho$ -topology if, and only if, for every  $x \in U$  there exists  $\varepsilon > 0$  such that  $B_\varepsilon(x) \subseteq U$ , where

$$(5.1) \quad B_\varepsilon(x) = \{y \in S \mid \rho(x, y) < \varepsilon\}.$$

Observe that without additional conditions (such that the triangle inequality for the function  $\rho$ ), we cannot assume that  $B_\varepsilon(x)$  are neighbourhood of  $x$ . When  $\{B_\varepsilon(x) \mid \varepsilon > 0\}$  is a neighbourhood base at  $x$  we say that  $S$  is semimetrizable and  $\rho$  is a semimetric.

Now we make precise what we mean by denting point with respect to  $\rho$ .

**Definition 5.3.** Let  $X$  be a normed space,  $F \subseteq X^*$  a norming subspace and  $\rho$  a symmetric on  $X$ . We say that  $x \in S_X$  is a  $\sigma(X, F)$ -denting point with respect to  $\rho$ , if for every  $\varepsilon > 0$  there exists a  $\sigma(X, F)$ -open half-space  $H$  such that  $x \in H$  and

$$\rho\text{-diam}(H \cap \mathcal{B}_X) := \sup \{\rho(y, z) \mid y, z \in H \cap \mathcal{B}_X\} < \varepsilon.$$

In the following lemma we define a natural symmetric on a rotund normed space.

**Lemma 5.4.** *Let  $X$  a normed space and  $F$  a norming subspace in  $X^*$ . Consider the function*

$$\rho(x, y) := 2\|x\|^2 + 2\|y\|^2 - \|x + y\|^2,$$

*defined for  $x, y \in X$ . The following holds:*

- (1)  $\rho$  is a non-negative function;
- (2) for every  $x \in \mathcal{S}_X$  and  $\varepsilon > 0$  there exists a  $\sigma(X, F)$ -open halfspace  $H$  such that  $\rho\text{-diam}(H \cap \mathcal{B}_X) < \varepsilon$ ;
- (3) if  $\|\cdot\|$  is rotund, then  $\rho$  is a symmetric. Furthermore on  $\mathcal{S}_X$  the  $\sigma(X, F)$ -topology is finer than the  $\tau_\rho$ -topology;
- (4) if  $\|\cdot\|$  is  $\sigma(X, F)$ -LUR, then the  $\tau_\rho$ -topology is finer than the  $\sigma(X, F)$ -topology. Furthermore the two topologies agree on  $\mathcal{S}_X$ .

*Proof.* (1) follows by the following inequality (see [DGZ93, Fact II.2.3]):

$$\rho(x, y) = 2\|x\|^2 + 2\|y\|^2 - \|x + y\|^2 \geq (\|x\| - \|y\|)^2 \geq 0.$$

- (2) Let  $\varepsilon > 0$  and  $x \in \mathcal{S}_X$ . Consider  $\mu \in (0, 1)$  and  $f \in \mathcal{B}_{X^*} \cap F$  such that  $4(2\mu - \mu^2) < \varepsilon$  and

$$x \in H_{f, \mu} := \{y \in X \mid f(y) > 1 - \mu\}.$$

For any  $y, z \in H_{f, \mu} \cap \mathcal{B}_X$  we have

$$\rho(y, z) = 2\|y\|^2 + 2\|z\|^2 - \|y + z\|^2 \leq 4 - (f(y + z))^2 \leq 4(2\mu - \mu^2) < \varepsilon.$$

So  $\rho\text{-diam}(H_{f, \mu} \cap \mathcal{B}_X) < \varepsilon$ .

- (3) By [DGZ93, Fact II.2.3] follows that  $\rho$  is a symmetric. The furthermore part follows by the  $\sigma(X, F)$ -lower semicontinuity of the norm and the equality

$$\{y \in \mathcal{S}_X \mid \rho(x, y) < \varepsilon\} = \left\{y \in \mathcal{S}_X \mid \left\| \frac{x + y}{2} \right\| > \sqrt{1 - \frac{\varepsilon}{4}}\right\},$$

which holds for any  $x \in \mathcal{S}_X$ .

- (4) Let  $W$  a  $\sigma(X, F)$ -open set. By contradiction, assume that there exists  $x \in W$  such that

$$B_\varepsilon(x) \cap (X \setminus W) \neq \emptyset,$$

for every  $\varepsilon > 0$ . Consider  $y_n \in B_{1/n}(x) \cap (X \setminus W)$  and observe that

$$\rho(x, y_n) = 2\|x\|^2 + 2\|y_n\|^2 - \|x + y_n\|^2 < \frac{1}{n}.$$

So  $\sigma(X, F)\text{-}\lim_{n \rightarrow +\infty} y_n = x$ , a contradiction.

□

We can now prove Theorem 1.6.

*Proof of Theorem 1.6.* Let  $\mathcal{H}$  be the family of  $\sigma(X, F)$ -open halfspace of  $X$ . For every  $k \in \mathbb{N}$  let  $\|\cdot\|_k$  be the norm obtained applying Theorem 1.3 to the set  $\mathcal{B}_X$  and the family

$$\mathcal{H}_k = \left\{ H \cap \mathcal{B}_X \mid H \in \mathcal{H}, H \cap \mathcal{B}_X \neq \emptyset \text{ and } \rho\text{-diam}(\Phi(H \cap \mathcal{B}_X)) < \frac{1}{k} \right\}.$$

Consider the equivalent  $\sigma(X, F)$ -lower semicontinuous norm

$$\|z\|_\Phi^2 = \|z\|^2 + \sum_{k \in \mathbb{N}} c_k \|z\|_k^2 \quad \text{for } z \in X,$$

where  $c_k$  are positive constants taken in such a way that the series converges uniformly on bounded subsets of  $X$ . Let  $x \in \mathcal{S}_X$  and  $(x_n)_{n \in \mathbb{N}} \subseteq \mathcal{B}_X$  such that

$$\lim_{n \rightarrow +\infty} (2\|x\|_\Phi^2 + 2\|x_n\|_\Phi^2 - \|x + x_n\|_\Phi^2) = 0.$$

By a standard convexity argument (see [DGZ93, Fact II.2.3]), for every  $k \in \mathbb{N}$  we have

$$\lim_{n \rightarrow +\infty} (2\|x\|_k^2 + 2\|x_n\|_k^2 - \|x + x_n\|_k^2) = 0.$$

By the thesis of Theorem 1.3, for every  $k \in \mathbb{N}$  there exists  $(H_n^k \cap \mathcal{B}_X)_{n \in \mathbb{N}} \subseteq \mathcal{H}_k$  and  $n_k \in \mathbb{N}$  with

$$(5.2) \quad x, x_n \in H_n^k \cap \mathcal{B}_X \quad \text{for every } n \geq n_k.$$

So  $\lim_{n \rightarrow +\infty} \rho(\Phi(x_n), \Phi(x)) = 0$ . By [Gru84, Lemma 9.3] we have  $\tau_\rho\text{-}\lim_{n \rightarrow +\infty} \Phi(x_n) = \Phi(x)$ . If  $x_n = y$  in (5.2), then  $\Phi(x) = \Phi(y)$ .  $\square$

Theorem 1.6 make it possible to construct equivalent norms satisfying some rotundity condition.

**Corollary 5.5.** *Let  $X$  a normed space and  $F$  a norming subspace in  $X^*$ .  $X$  admits an equivalent  $\sigma(X, F)$ -lower semicontinuous rotund norm if, and only if, a symmetric  $\rho$  and an equivalent  $\sigma(X, F)$ -lower semicontinuous norm  $\|\cdot\|_\rho$  exist such that every point of its unit sphere is a  $F$ -denting point with respect to  $\rho$ .*

*Proof.* The only if part is Lemma 5.4. Let  $\|\cdot\|_\Phi$  the equivalent and  $\sigma(X, F)$ -lower semicontinuous norm obtained applying Theorem 1.6 with  $\Phi = id$ . Consider the equivalent  $\sigma(X, F)$ -lower semicontinuous norm

$$\|z\|_R^2 = \|z\|_\rho^2 + \|z\|_\Phi^2 \quad \text{for } z \in X.$$

Let's now prove rotundness. Let  $x, y \in X$  with

$$2\|x\|_R^2 + 2\|y\|_R^2 - \|x + y\|_R^2 = 0.$$

By a standard convexity argument (see [DGZ93, Fact II.2.3]) we have

$$(5.3) \quad 2\|x\|_\rho^2 + 2\|y\|_\rho^2 - \|x + y\|_\rho^2 = 0;$$

$$(5.4) \quad 2\|x\|_\Phi^2 + 2\|y\|_\Phi^2 - \|x + y\|_\Phi^2 = 0.$$

(5.3) and [DGZ93, Fact II.2.3] give us  $\|x\|_\rho = \|y\|_\rho$ . Divide (5.4) by  $\|x\|_\rho^2 = \|y\|_\rho^2$  obtaining

$$2 \left\| \frac{x}{\|x\|_\rho} \right\|_\Phi^2 + 2 \left\| \frac{y}{\|y\|_\rho} \right\|_\Phi^2 - \left\| \frac{x}{\|x\|_\rho} + \frac{y}{\|y\|_\rho} \right\|_\Phi^2 = 0.$$

By the thesis of Theorem 1.6 we have  $\Phi(x/\|x\|_\rho) = \Phi(y/\|y\|_\rho)$ , which means  $x = y$ .  $\square$

Via a suitable use of Theorem 1.5 we can improve our result in the dual case.

**Corollary 5.6.** *Let  $X^*$  be a dual Banach space.  $X^*$  admits an equivalent, dual rotund norm if, and only if, there exists a symmetric  $\rho$  on  $X^*$  and an equivalent dual norm  $\|\cdot\|_\rho$  such that for every  $x^*$  on the unit sphere of  $\|\cdot\|_\rho$  and  $\varepsilon > 0$  there exists a  $w^*$ -neighbourhood  $U$  of  $x^*$  such that*

$$\rho\text{-diam}(U \cap B_\rho) < \varepsilon,$$

where  $B_\rho = \{x^* \in X^* \mid \|x^*\|_\rho \leq 1\}$ . This means that every point of the unit sphere of  $\|\cdot\|_\rho$  has a  $w^*$ -neighbourhood, relatively to the unit ball  $B_\rho$ , of arbitrarily small  $\rho$ -diameter.

*Proof.* Define  $B = \{x^* \in X^* \mid \|x^*\|_\rho \leq 1\}$  and  $S = \{x^* \in X^* \mid \|x^*\|_\rho = 1\}$ . Let  $\mathcal{U}$  be the family of  $w^*$ -open sets of  $X^*$ . Consider the countable collection of covers of  $S$

$$\mathcal{U}_n = \left\{ U \cap B \mid U \in \mathcal{U}, U \cap B_X \neq \emptyset \text{ and } \rho\text{-diam}(U \cap B) < \frac{1}{n} \right\}.$$

It is easy to see that this family is a  $G_\delta$ -diagonal sequence for  $S$ , and by Theorem 1.5 the thesis follows.  $\square$

If we add some request on the relation between the topology generated by the symmetric and the  $\sigma(X, F)$ -topology then we can obtain a locally uniformly rotund norm.

**Corollary 5.7.** *Let  $X$  a normed space and  $F$  a norming subspace in  $X^*$ .  $X$  admits an equivalent  $\sigma(X, F)$ -lower semicontinuous and  $\sigma(X, F)$ -LUR norm if, and only if, a symmetric  $\rho$  and an equivalent  $\sigma(X, F)$ -lower semicontinuous norm  $\|\cdot\|_\rho$  exist such that the topology  $\tau_\rho$  is finer than the  $\sigma(X, F)$ -topology and every point of its unit sphere  $S_\rho$  is a  $F$ -denting point with respect to  $\rho$ .*

*Proof.* The only if part is Lemma 5.4. Let  $\|\cdot\|_\Phi$  be the equivalent and  $\sigma(X, F)$ -lower semicontinuous norm obtained applying Theorem 1.6 with  $\Phi = id$ . Consider the equivalent  $\sigma(X, F)$ -lower semicontinuous norm

$$\|z\|_L^2 = \|z\|_\rho^2 + \|z\|_\Phi^2 \quad \text{for } z \in X.$$

Let's now prove the  $\sigma(X, F)$ -LUR property. Let  $(x_n)_{n \in \mathbb{N}} \subseteq X$  and  $x \in X$  with

$$\lim_{n \rightarrow +\infty} (2\|x\|_L^2 + 2\|x_n\|_L^2 - \|x + x_n\|_L^2) = 0.$$

If  $x = 0$ , there is nothing to prove. Assume without loss of generality that  $x \neq 0$  and  $x_n \neq 0$  for every  $n \in \mathbb{N}$ . By a standard convexity argument (see [DGZ93, Fact II.2.3]) we have

$$(5.5) \quad \lim_{n \rightarrow +\infty} \|x_n\|_\rho = \|x\|_\rho.$$

By Lemma 2.2 we have

$$\lim_{n \rightarrow +\infty} \left( 2 \left\| \frac{x}{\|x\|_\rho} \right\|_L^2 + 2 \left\| \frac{x_n}{\|x_n\|_\rho} \right\|_L^2 - \left\| \frac{x}{\|x\|_\rho} + \frac{x_n}{\|x_n\|_\rho} \right\|_L^2 \right) = 0.$$

By a standard convexity argument (see [DGZ93, Fact II.2.3])

$$(5.6) \quad \lim_{n \rightarrow +\infty} \left( 2 \left\| \frac{x}{\|x\|_\rho} \right\|_\Phi^2 + 2 \left\| \frac{x_n}{\|x_n\|_\rho} \right\|_\Phi^2 - \left\| \frac{x}{\|x\|_\rho} + \frac{x_n}{\|x_n\|_\rho} \right\|_\Phi^2 \right) = 0.$$

By the thesis of Theorem 1.6 we have that  $\tau_\rho\text{-}\lim_{n \rightarrow +\infty} x_n / \|x_n\|_\rho = x / \|x\|_\rho$ . Since the  $\tau_\rho$ -topology is finer than the  $\sigma(X, F)$ -topology we obtain

$$\sigma(X, F)\text{-}\lim_{n \rightarrow +\infty} \frac{x_n}{\|x_n\|_\rho} = \frac{x}{\|x\|_\rho}.$$

The thesis follows by equality (5.5).  $\square$

Using Theorem 1.3 we can improve the previous corollary in the dual case.

**Theorem 5.8.** *Let  $X^*$  a dual Banach space.  $X^*$  admits an equivalent dual  $w^*$ -LUR norm if, and only if, an equivalent dual norm  $\|\cdot\|_M$  exists such that the  $w^*$ -topology is semimetrizable when restricted to its unit sphere.*

*Proof.* The only if part follows by [MOTV99, Proposition 4] and the fact that every Moore space is semimetrizable (see [SS95, Pag. 164]). Without loss of generality we can assume that the  $w^*$ -topology is semimetrizable when restricted to the unit sphere  $\mathcal{S}_{X^*}$ . Let  $\mathcal{U}$  be the family of  $w^*$ -open subset of  $\mathcal{B}_{X^*}$ . Consider the family

$$\mathcal{U}_+ = \left\{ U \in \mathcal{U} \mid \text{there exists } n \in \mathbb{N} \text{ and } f_i \in X \text{ and } \alpha_i \geq 0 \text{ for } i = 1, \dots, n \text{ with } U = \bigcap_{i=1}^n \{x^* \in \mathcal{B}_{X^*} \mid f_i(x^*) > \alpha_i\} \right\}.$$

Let  $x^* \in \mathcal{S}_{X^*}$  and  $W \in \mathcal{U}$  such that  $x^* \in W$ . It is known that there exist  $n \in \mathbb{N}$ ,  $\alpha_i \in \mathbb{R}$  and  $f_i \in \mathcal{S}_X$  for  $i = 1, \dots, n$  such that

$$x^* \in W_1 := \bigcap_{i=1}^n \{y^* \in \mathcal{B}_{X^*} \mid f_i(y^*) > \alpha_i\} \subseteq W.$$

We claim that there exists  $W_2 \in \mathcal{U}_+$  with  $x^* \in W_2 \subseteq W_1$ . Indeed, consider

$$H_k := \{y^* \in \mathcal{B}_{X^*} \mid f_k(y^*) > \alpha_k\}$$

for every  $k \in I_- = \{i = 1, \dots, n \mid \alpha_i < 0\}$ . By the Hahn–Banach theorem there exists  $f'_k \in X$  and  $\alpha'_k$  such that

$$f'_k(x^*) > \alpha'_k \text{ and } f'_k(y^*) \leq \alpha'_k \text{ for every } y^* \in \text{conv}((\mathcal{B}_{X^*} \setminus H_k) \cup \{0\}).$$

Observe that  $\alpha'_k \geq 0$ , in fact if  $\alpha'_k < 0$  then  $f'_k(0) \leq \alpha'_k < 0$ . Then

$$x^* \in W_2 := \bigcap_{i=1}^n \{y^* \in \mathcal{B}_{X^*} \mid f'_i(y^*) > \alpha'_i\} \subseteq W_1,$$

and  $W_2 \in \mathcal{U}_+$ .

For every  $n \in \mathbb{N}$  consider the family

$$\mathcal{U}_n = \left\{ W \in \mathcal{U} \mid \text{there exists } W_1 \in \mathcal{U}_+ \text{ with } \overline{W}^{w^*} \subseteq W_1, W \cap \mathcal{S}_{X^*} \neq \emptyset, \rho\text{-diam}(W_1 \cap \mathcal{S}_{X^*}) < \frac{1}{n} \right\}.$$

Observe that every  $\mathcal{U}_n$  is a cover of  $\mathcal{S}_{X^*}$ . Using the same notation of Lemma 3.3 we have

$$B^{(\omega_0)}(\mathcal{B}_{X^*}, \mathcal{U}_n) \cap \mathcal{S}_{X^*} = \emptyset.$$



Let  $B_n^{(m)} := B^{(m)}(\mathcal{B}_{X^*}, \mathcal{U}_n)$ . For every  $n, m \in \mathbb{N}$  let  $\|\cdot\|_{n,m}$  the equivalent dual norm obtained applying Theorem 1.3 with  $B_n^{(m)}$  and the family  $\text{Sl}_X(B_n^{(m)}, \mathcal{U}_n)$ . Consider the equivalent dual norm

$$\|z^*\|_L^2 := \|z^*\|^2 + \sum_{n,m \in \mathbb{N}} c_{n,m} \|z^*\|_{n,m}^2 \quad \text{for } z^* \in X^*.$$

Now we want to prove that  $\|\cdot\|_L$  is  $w^*$ -LUR. Let  $x^* \in X^*$  and  $(x_k^*)_{k \in \mathbb{N}} \subseteq X^*$  such that

$$\lim_{k \rightarrow +\infty} (2\|x^*\|_L^2 + 2\|x_k^*\|_L^2 - \|x^* + x_k^*\|_L^2) = 0.$$

If  $x^* = 0$ , there is nothing to prove. Assume that  $x^* \neq 0$  and, without loss of generality,  $x_k^* \neq 0$ . By a standard convexity argument (see [DGZ93, Fact II.2.3]) we have

$$\lim_{k \rightarrow +\infty} \|x_k^*\| = \|x^*\|,$$

so, for every  $n, m \in \mathbb{N}$ , by Lemma 2.2 we have

$$(\dagger_{n,m}) \quad \lim_{k \rightarrow +\infty} \left( 2 \left\| \frac{x^*}{\|x^*\|} \right\|_{n,m}^2 + 2 \left\| \frac{x_k^*}{\|x_k^*\|} \right\|_{n,m}^2 - \left\| \frac{x^*}{\|x^*\|} + \frac{x_k^*}{\|x_k^*\|} \right\|_{n,m}^2 \right) = 0.$$

For every  $z \in \mathcal{S}_{X^*}$  let

$$m_n(z^*) := \min\{m \in \mathbb{N} \mid z^* \in B_n^{(m)} \text{ and } z^* \notin B_n^{(m+1)}\},$$

such quantity is well defined by Lemma 3.3. Let  $m_n := m_n(x^*/\|x^*\|)$ . By Theorem 1.3 and  $(\dagger_{n,m_n})$  we have that there exists  $(H_k^n \cap B_n^{(m_n)})_{k \in \mathbb{N}} \subseteq \text{Sl}_X(B_n^{(m_n)}, \mathcal{U}_n)$  such that for every  $\delta > 0$  there exists  $k_\delta \in \mathbb{N}$  with  $x^*/\|x^*\| \in H_k^n \cap B_n^{(m_n)}$  and

$$\frac{x_k^*}{\|x_k^*\|} \in \overline{(H_k^n \cap B_n^{(m_n)}) + \delta \mathcal{B}_{X^*}}^{w^*} \quad \text{for } k \geq k_\delta.$$

By definition there exists  $W_k^n \in \mathcal{U}_n$  such that  $x^*/\|x^*\| \in W_k^n$  and

$$\frac{x_k^*}{\|x_k^*\|} \in \overline{W_k^n + \delta \mathcal{B}_{X^*}}^{w^*} \quad \text{for } k \geq k_\delta.$$

Furthermore there exists  $W_{k,1}^n \in \mathcal{U}_+$  with  $\overline{W_k^n}^{w^*} \subseteq W_{k,1}^n$ ,  $W_k^n \cap \mathcal{S}_{X^*} \neq \emptyset$  and

$$\rho\text{-diam}(W_{k,1}^n \cap \mathcal{S}_{X^*}) < 1/n.$$

We have  $x^*/\|x^*\| \in W_{k,1}^n \cap \mathcal{S}_{X^*}$  and for  $k \geq k_\delta$ . Indeed

$$\frac{x_k^*}{\|x_k^*\|} \in (\overline{W_k^n + \delta \mathcal{B}_{X^*}}^{w^*}) \cap \mathcal{S}_{X^*} \subseteq_{(i)} (\overline{W_k^n}^{w^*} + \delta \overline{\mathcal{B}_{X^*}}^{w^*}) \cap \mathcal{S}_{X^*} \subseteq (W_{k,1}^n + \delta \overline{\mathcal{B}_{X^*}}^{w^*}) \cap \mathcal{S}_{X^*},$$

where (i) follows by compactness. We claim that

$$(W_{k,1}^n + \delta \overline{\mathcal{B}_{X^*}}^{w^*}) \cap \mathcal{S}_{X^*} \subseteq (W_{k,1}^n \cap \mathcal{S}_{X^*}) + (2\delta) \overline{\mathcal{B}_{X^*}}^{w^*}.$$

Indeed if  $z^* \in (W_{k,1}^n + \delta \overline{\mathcal{B}_{X^*}}^{w^*}) \cap \mathcal{S}_{X^*}$ , then  $z^* = w^* + t^*$  with  $w^* \in W_{k,1}^n$ ,  $t^* \in \delta \overline{\mathcal{B}_{X^*}}^{w^*}$  and  $\|z^*\| = 1$ . We have

$$1 = \|z^*\| \leq \|w^*\| + \|t^*\| \leq \|w^*\| + \delta \implies 1 - \delta \leq \|w^*\| \leq 1.$$

Observe that  $w^*/\|w^*\| \in W_{k,1}^n \cap \mathcal{S}_{X^*}$ , by the very definition of  $\mathcal{U}_+$ , and

$$\left\| \left( 1 - \frac{1}{\|w^*\|} \right) w^* + t^* \right\| \leq \left| 1 - \frac{1}{\|w^*\|} \right| \|w^*\| + \|t^*\| \leq (1 - \|w^*\|) + \delta \leq 2\delta.$$

To conclude the prove of our claim observe that

$$z^* = \frac{w^*}{\|w^*\|} + \left( 1 - \frac{1}{\|w^*\|} \right) w^* + t^* \in (W_{k,1}^n \cap \mathcal{S}_{X^*}) + (2\delta) \overline{\mathcal{B}_{X^*}}^{w^*}.$$

Now for every  $n \in \mathbb{N}$  we know that there exists  $k_{1/n} \in \mathbb{N}$  such that for every  $k \geq k_{1/n}$  we have  $x^*/\|x^*\| \in W_{k,1}^n \cap \mathcal{S}_{X^*}$  and

$$\frac{x_k^*}{\|x_k^*\|} \in (W_{k,1}^n \cap \mathcal{S}_{X^*}) + \frac{2}{n} \overline{\mathcal{B}_{X^*}}^{w^*}.$$

Let  $w_k^n \in W_{k,1}^n \cap \mathcal{S}_{X^*}$  and  $t_k^n \in (2/n) \overline{\mathcal{B}_{X^*}}^{w^*}$  such that for every  $n \in \mathbb{N}$  and  $k \geq k_{1/n}$  we have

$$\frac{x_k^*}{\|x_k^*\|} = w_k^n + t_k^n.$$

We know that  $\rho(w_k^n, x^*/\|x^*\|) < 1/n$ . Without loss of generality we can assume that for every  $n \in \mathbb{N}$  that  $k_{1/n} < k_{1/(n+1)}$  and let

$$W_k := w_k^n \text{ and } T_k := t_k^n \text{ for every } n \in \mathbb{N} \text{ and } k_{1/n} \leq k < k_{1/(n+1)}.$$

In particular we have that

$$\frac{x_k^*}{\|x_k^*\|} = W_k + T_k \text{ for } k \geq k_1 \text{ and } \lim_{k \rightarrow +\infty} \rho\left(W_k, \frac{x^*}{\|x^*\|}\right) = 0.$$

By [Gru84, Lemma 9.3] it follows that  $w^*$ - $\lim_{k \rightarrow +\infty} W_k = x^*/\|x^*\|$ . We claim that  $w^*$ - $\lim_{k \rightarrow +\infty} x_k^* = x^*$ . In fact, let  $f \in X$ ,  $\varepsilon > 0$  and choose  $k_0 \in \mathbb{N}$  such that for every  $k \geq k_0$

$$|\|x^*\| - \|x_k^*\|| < \frac{\varepsilon}{3}, \left| f\left(W_k - \frac{x^*}{\|x^*\|}\right) \right| < \frac{\varepsilon}{3\|x^*\|} \text{ and } \|T_k\| \leq \frac{\varepsilon}{3\|x^*\|\|f\|}.$$

The following easy calculation gives our claim

$$\begin{aligned} |f(x_k^* - x^*)| &= \|x^*\| \left| f\left(\frac{x_k^*}{\|x_k^*\|} - \frac{x^*}{\|x^*\|}\right) \right| \leq \|x^*\| \left( \left| f\left(\frac{x_k^*}{\|x_k^*\|} - \frac{x_k^*}{\|x^*\|}\right) \right| + \left| f\left(\frac{x_k^*}{\|x_k^*\|} - \frac{x^*}{\|x^*\|}\right) \right| \right) \leq \\ &\leq \|x^*\| \left( \left| 1 - \frac{\|x_k^*\|}{\|x^*\|} \right| + \left| f\left(W_k - \frac{x^*}{\|x^*\|}\right) \right| + |f(T_k)| \right) \leq \\ &\leq |\|x_k^*\| - \|x^*\|| + \|x^*\| \left| f\left(W_k - \frac{x^*}{\|x^*\|}\right) \right| + \|x^*\| \|f\| \|T_k\| < \varepsilon. \end{aligned}$$

□

The next result was already proved in a different way in [FOR16].

**Theorem 5.9.** *Let  $X$  a normed space and  $F$  a norming subspace in  $X^*$ . The following fact are equivalent:*

- (1) *there exists an equivalent norm  $\|\cdot\|_A$  such that for every  $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \subseteq X$  and  $x \in X$  if*

$$\lim_{n \rightarrow +\infty} (2\|x_n\|_A^2 + 2\|y_n\|_A^2 - \|x_n + y_n\|_A^2) = \lim_{n \rightarrow +\infty} (2\|x\|_A^2 + 2\|x_n\|_A^2 - \|x + x_n\|_A^2) = 0,$$

*then  $\sigma(X, F)\text{-}\lim_{n \rightarrow +\infty} y_n = x$ ;*

- (2) *An equivalent  $\sigma(X, F)$ -lower semicontinuous norm  $\|\cdot\|_M$  exists such that  $(X, \sigma(X, F))$  admits a finer metric topology  $(X, d)$  such that every point of its unit sphere is a  $F$ -denting point with respect to  $d$ .*

*Proof.* Let's start with  $(1 \Rightarrow 2)$ .  $\|\cdot\|_A$  is obviously  $\sigma(X, F)$ -LUR, so by Lemma 5.4 the symmetric

$$\rho(x, y) = 2\|x\|_A^2 + 2\|y\|_A^2 - \|x + y\|_A^2$$

define a finer topology than the  $\sigma(X, F)$ -topology. By [Gru84, Theorem 9.14] the  $\tau_\rho$ -topology is metrizable by a metric  $d$ . By Lemma 5.4 for every  $x \in S_A$  and  $\varepsilon > 0$  there exists a  $\sigma(X, F)$ -open halfspace  $H$  such that

$$\rho\text{-diam}(H \cap B_A) < \varepsilon,$$

where  $S_A = \{x \in X \mid \|x\|_A = 1\}$  and  $B_A = \{x \in X \mid \|x\|_A \leq 1\}$ . This condition is equivalent to ask

$$H \cap B_A \subseteq B_\varepsilon^\rho(x) := \{y \in X \mid \rho(x, y) < \varepsilon\}.$$

Since  $d$  generates the same topology of  $\rho$ , then for every  $\eta > 0$  and  $x \in X$  there exists  $\varepsilon(\eta) > 0$  such that

$$B_{\varepsilon(\eta)}^\rho(x) \subseteq B_\eta^d(x) := \{y \in X \mid d(x, y) < \eta\}.$$

This last two argument gives the thesis. For the other implication let  $\mathcal{H}$  the family of  $\sigma(X, F)$ -open halfspaces of  $X$ . For each  $k \in \mathbb{N}$  consider the family

$$\mathcal{H}_k = \left\{ H \cap B_M \mid H \in \mathcal{H}, H \cap B_M \neq \emptyset \text{ and } d\text{-diam}(H \cap B_M) < \frac{1}{k} \right\}.$$

Let  $\|\cdot\|_k$  the norm obtained applying Theorem 1.4 to  $B_M$  and the family  $\mathcal{H}_k$ . Consider the  $\sigma(X, F)$ -lower semicontinuous norm

$$\|z\|_A^2 = \|z\|_M^2 + \sum_{k \in \mathbb{N}} c_k \|z\|_k^2 \quad \text{for } z \in X,$$

where  $(c_k)_{k \in \mathbb{N}}$  are positive constants taken in such a way that the series converges uniformly on bounded subsets of  $X$ .

Let  $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \subseteq X$  and  $x \in X$  if

$$\lim_{n \rightarrow +\infty} (2\|x_n\|_A^2 + 2\|y_n\|_A^2 - \|x_n + y_n\|_A^2) = \lim_{n \rightarrow +\infty} (2\|x\|_A^2 + 2\|x_n\|_A^2 - \|x + x_n\|_A^2) = 0.$$

If  $x = 0$  there is nothing to prove. We can assume that  $x \neq 0$  and for every  $n \in \mathbb{N}$   $x_n \neq 0$  and  $y_n \neq 0$ . By a standard convexity argument (see [DGZ93, Fact II.2.3]) we know

$\lim_{n \rightarrow +\infty} \|x_n\|_M = \|x\|_M$ , applying Lemma 2.2 we obtain

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \left( 2 \left\| \frac{x_n}{\|x_n\|_M} \right\|_A^2 + 2 \left\| \frac{y_n}{\|y_n\|_M} \right\|_A^2 - \left\| \frac{x_n}{\|x_n\|_M} + \frac{y_n}{\|y_n\|_M} \right\|_A^2 \right) = \\ & = \lim_{n \rightarrow +\infty} \left( 2 \left\| \frac{x}{\|x\|_M} \right\|_A^2 + 2 \left\| \frac{x_n}{\|x_n\|_M} \right\|_A^2 - \left\| \frac{x}{\|x\|_M} + \frac{x_n}{\|x_n\|_M} \right\|_A^2 \right) = 0. \end{aligned}$$

By Lemma 2.1, for every  $k \in \mathbb{N}$ , we have

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \left( 2 \left\| \frac{x_n}{\|x_n\|_M} \right\|_k^2 + 2 \left\| \frac{y_n}{\|y_n\|_M} \right\|_k^2 - \left\| \frac{x_n}{\|x_n\|_M} + \frac{y_n}{\|y_n\|_M} \right\|_k^2 \right) = \\ & = \lim_{n \rightarrow +\infty} \left( 2 \left\| \frac{x}{\|x\|_M} \right\|_k^2 + 2 \left\| \frac{x_n}{\|x_n\|_M} \right\|_k^2 - \left\| \frac{x}{\|x\|_M} + \frac{x_n}{\|x_n\|_M} \right\|_k^2 \right) = 0. \end{aligned}$$

By the thesis of Theorem 1.4 we obtain that for every  $k \in \mathbb{N}$  there exist two sequences  $(H_n^{L,k})_{n \in \mathbb{N}}, (H_n^{U,k})_{n \in \mathbb{N}} \subseteq \mathcal{H}_k$  such that

- (i)  $n_{L,k} \in \mathbb{N}$  exists with  $x/\|x\|_M, x_n/\|x_n\|_M \in H_n^{L,k}$  for  $n \geq n_{L,k}$ ;
- (ii)  $n_{U,k} \in \mathbb{N}$  exists with

$$\frac{x_n}{\|x_n\|_M}, \frac{y_n}{\|y_n\|_M} \in H_n^{U,k} \quad \text{for } n \geq n_{U,k}.$$

By this two conditions follows

$$d\left(\frac{x}{\|x\|_M}, \frac{x_n}{\|x_n\|_M}\right) < \frac{1}{k} \text{ for } n \geq n_{L,k} \text{ and } d\left(\frac{x_n}{\|x_n\|_M}, \frac{y_n}{\|y_n\|_M}\right) < \frac{1}{k} \text{ for } n \geq n_{U,k}.$$

So we have that

$$\lim_{n \rightarrow +\infty} d\left(\frac{x}{\|x\|_M}, \frac{x_n}{\|x_n\|_M}\right) = \lim_{n \rightarrow +\infty} d\left(\frac{x_n}{\|x_n\|_M}, \frac{y_n}{\|y_n\|_M}\right) = 0,$$

and by the triangle inequality we obtain

$$\lim_{n \rightarrow +\infty} d\left(\frac{x}{\|x\|_M}, \frac{y_n}{\|y_n\|_M}\right) = 0.$$

Since  $d$  generates a topology finer than the  $\sigma(X, F)$ -topology we obtain

$$\sigma(X, F)\text{-}\lim_{n \rightarrow +\infty} \frac{y_n}{\|y_n\|_M} = \frac{x}{\|x\|_M}.$$

By Lemma 2.3 we have  $\sigma(X, F)\text{-}\lim_{n \rightarrow +\infty} y_n = x$ . □

Using Theorem 1.4 it is possible to prove a stronger version of the previous result in the dual case. The proof is essentially the same as the prove of Theorem 5.8, with obviuos modifications. We leave it to the reader. A different proof can be found in [FOR16].

**Theorem 5.10.** *Let  $X^*$  a dual Banach space.  $X^*$  admits an equivalent dual norm  $\|\cdot\|_A$  with the following property: for every  $(x_n^*)_{n \in \mathbb{N}}, (y_n^*)_{n \in \mathbb{N}} \subseteq X^*$  and  $x^* \in X^*$  if*

$$\lim_{n \rightarrow +\infty} (2\|x_n^*\|_A^2 + 2\|y_n^*\|_A^2 - \|x_n^* + y_n^*\|_A^2) = \lim_{n \rightarrow +\infty} (2\|x^*\|_A^2 + 2\|x_n^*\|_A^2 - \|x^* + x_n^*\|_A^2) = 0,$$

then  $w^*$ - $\lim_{n \rightarrow +\infty} y_n = x$ , if, and only if, an equivalent dual norm  $\|\cdot\|_M$  exists such that the  $w^*$ -topology is metrizable when restricted to its unit sphere.

## 6. A SLICE VERSION OF MOORE SPACES

Recall the following definition (see [Gru84, Definition 1.3]).

**Definition 6.1.** A topological space  $X$  is developable if there exists a sequence  $(\mathcal{G}_n)_{n \in \mathbb{N}}$  of open covers of  $X$  such that for each  $x \in X$ ,

$$\{\text{st}(x, \mathcal{G}_n) \mid n \in \mathbb{N}\}$$

is a neighbourhood base at  $x$ . Furthermore, if  $X$  is a regular space, we say that  $X$  is a Moore space.

Any sequence  $(\mathcal{G}_n)_{n \in \mathbb{N}}$  satisfying the conditions of definition 6.1 is said to be a development. In addition if  $X$  is a subset of a topological vector space and  $(\mathcal{G}_n)_{n \in \mathbb{N}}$  a development of  $X$ , with the property that every element of  $\bigcup_{n \in \mathbb{N}} \mathcal{G}_n$  is a  $\sigma(X, F)$ -open slice of  $X$ , we say that  $X$  is a developable (respectively, Moore) space with  $\sigma(X, F)$ -slices.

In [MOTV99, Proposition 4] was proved that the unit sphere of a  $w$ -LUR norm is a Moore space with  $w$ -slices. Actually the proof can be adapted to show that the unit sphere of a  $\sigma(X, F)$ -lower semicontinuous and  $\sigma(X, F)$ -LUR norm is a Moore space with  $\sigma(X, F)$ -slices. The following result is a converse of what just said.

*Proof.* (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) can be found in [MOTV99, Proposition 4], while (3) $\Rightarrow$ (1) follows by the fact that a Moore space is a semimetric space (see [SS95, Pag. 164]) and Theorem 5.8. Let  $(\mathcal{H}_n)_{n \in \mathbb{N}}$  a development for the unit sphere  $S_M$  of the norm  $\|\cdot\|_M$ , such that every element of  $\bigcup_{n \in \mathbb{N}} \mathcal{H}_n$  is a  $\sigma(X, F)$ -slice for every  $n \in \mathbb{N}$ . Apply Theorem 1.3 with the set  $S_M$  and the family  $\mathcal{H}_n$  in order to obtain a norm  $\|\cdot\|_L$ . Consider the equivalent and  $\sigma(X, F)$ -lower semicontinuous norm

$$\|z\|_L^2 = \|z\|_M^2 + \sum_{n \in \mathbb{N}} c_n \|z\|_n^2 \quad \text{for } z \in X,$$

where  $c_n$  are positive constants taken in such a way that the series converges uniformly on bounded subsets of  $X$ .

We claim that  $\|\cdot\|_L$  is  $\sigma(X, F)$ -LUR. Let  $x \in X$  and  $(x_k)_{k \in \mathbb{N}} \subseteq X$  such that

$$\lim_{k \rightarrow +\infty} (2\|x\|_L^2 + 2\|x_k\|_L^2 - \|x + x_k\|_L^2) = 0.$$

If  $x = 0$ , then there is nothing to prove. Assume that  $x \neq 0$  and, without loss of generality that  $x_k \neq 0$  for every  $k \in \mathbb{N}$ . By a standard convexity argument (see [DGZ93, Fact II.2.3]) we have

$$\lim_{k \rightarrow +\infty} \|x_k\|_M = \|x\|_M,$$

then by Lemma 2.2 we have

$$\lim_{k \rightarrow +\infty} \left( 2 \left\| \frac{x}{\|x\|_M} \right\|_L^2 + 2 \left\| \frac{x_k}{\|x_k\|_M} \right\|_L^2 - \left\| \frac{x}{\|x\|_M} + \frac{x_k}{\|x_k\|_M} \right\|_L^2 \right) = 0.$$

By a standard convexity argument (see [DGZ93, Fact II.2.3]) we have for every  $n \in \mathbb{N}$

$$\lim_{k \rightarrow +\infty} \left( 2 \left\| \frac{x}{\|x\|_M} \right\|_n^2 + 2 \left\| \frac{x_k}{\|x_k\|_M} \right\|_n^2 - \left\| \frac{x}{\|x\|_M} + \frac{x_k}{\|x_k\|_M} \right\|_n^2 \right) = 0.$$

By the thesis of Theorem 1.3 we have that for every  $n \in \mathbb{N}$  there exists a sequence  $(H_k^n)_{k \in \mathbb{N}} \subseteq \mathcal{H}_n$  and  $k_n \in \mathbb{N}$  such that

$$\frac{x}{\|x\|_M}, \frac{x_k}{\|x_k\|_M} \in H_k^n \cap S_M \quad \text{for } k \geq k_n.$$

This implies that for every  $n \in \mathbb{N}$  there exists  $k_n \in \mathbb{N}$  such that

$$\frac{x_k}{\|x_k\|_M} \in \text{st} \left( \frac{x}{\|x\|_M}, \mathcal{H}_n \right).$$

Since  $(S_M, \sigma(X, F))$  is a Moore space we have that

$$\sigma(X, F)\text{-}\lim_{k \rightarrow +\infty} \frac{x_k}{\|x_k\|_M} = \frac{x}{\|x\|_M}.$$

By Lemma 2.3 we get that  $\sigma(X, F)\text{-}\lim_{k \rightarrow +\infty} x_k = x$ . So  $\|\cdot\|_L$  is  $\sigma(X, F)$ -LUR.  $\square$

## REFERENCES

- [Ced61] J. G. Ceder. Some generalizations of metric spaces. *Pacific J. Math.*, 11:105–125, 1961.
- [Cho69] G. Choquet. *Lectures on analysis. Vol. II: Representation theory*. Edited by J. Marsden, T. Lance and S. Gelbart. W. A. Benjamin, Inc., New York-Amsterdam, 1969.
- [Cla36] J. A. Clarkson. Uniformly convex spaces. *Trans. Amer. Math. Soc.*, 40(3):396–414, 1936.
- [DGZ93] R. Deville, G. Godefroy, and V. Zizler. *Smoothness and renormings in Banach spaces*, volume 64 of *Pitman Monographs and Surveys in Pure and Applied Mathematics*. Longman Scientific & Technical, Harlow, 1993.
- [FHH<sup>+</sup>11] M. Fabian, P. Habala, P. Hájek, V. Montesinos, and V. Zizler. *Banach space theory*. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer, New York, 2011. The basis for linear and nonlinear analysis.
- [FOOR16] S. Ferrari, L. Oncina, J. Orihuela, and M. Raja. Metrization theory and the Kadec property. *Banach J. Math. Anal.*, 10(2):281–306, 2016.
- [FOR16] S. Ferrari, J. Orihuela, and M. Raja. Weakly metrizability of spheres and renormings of Banach spaces. *Q. J. Math.*, 67(1):15–27, 2016.
- [God01] G. Godefroy. Renormings of Banach spaces. In *Handbook of the geometry of Banach spaces, Vol. I*, pages 781–835. North-Holland, Amsterdam, 2001.
- [Gru84] G. Gruenhage. Generalized metric spaces. In *Handbook of set-theoretic topology*, pages 423–501. North-Holland, Amsterdam, 1984.
- [Lan93] G. Lancien. Dentability indices and locally uniformly convex renormings. *Rocky Mountain J. Math.*, 23(2):635–647, 1993.
- [Lan95] G. Lancien. On uniformly convex and uniformly Kadec-Klee renormings. *Serdica Math. J.*, 21(1):1–18, 1995.

- [MOTV99] A. Moltó, J. Orihuela, S. Troyanski, and M. Valdivia. On weakly locally uniformly rotund Banach spaces. *J. Funct. Anal.*, 163(2):252–271, 1999.
- [MOTZ07] A. Moltó, J. Orihuela, S. Troyanski, and V. Zizler. Strictly convex renormings. *J. Lond. Math. Soc. (2)*, 75(3):647–658, 2007.
- [OST12] J. Orihuela, R. J. Smith, and S. Troyanski. Strictly convex norms and topology. *Proc. Lond. Math. Soc. (3)*, 104(1):197–222, 2012.
- [OT09a] J. Orihuela and S. Troyanski. Deville’s master lemma and Stone’s discreteness in renorming theory. *J. Convex Anal.*, 16(3-4):959–972, 2009.
- [OT09b] J. Orihuela and S. Troyanski. LUR renormings through Deville’s master lemma. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM*, 103(1):75–85, 2009.
- [Raj99a] M. Raja. Kadec norms and Borel sets in a Banach space. *Studia Math.*, 136(1):1–16, 1999.
- [Raj99b] M. Raja. Locally uniformly rotund norms. *Mathematika*, 46(2):343–358, 1999.
- [Raj02] M. Raja. On dual locally uniformly rotund norms. *Israel J. Math.*, 129:77–91, 2002.
- [Raj07] M. Raja. Dentability indices with respect to measures of non-compactness. *J. Funct. Anal.*, 253(1):273–286, 2007.
- [Raj13] M. Raja. On asymptotically uniformly smooth Banach spaces. *J. Funct. Anal.*, 264(2):479–492, 2013.
- [Šmu40] V. Šmulian. Sur la dérivabilité de la norme dans l’espace de Banach. *C. R. (Doklady) Acad. Sci. URSS (N. S.)*, 27:643–648, 1940.
- [SS95] L. A. Steen and J. A. J. Seebach. *Counterexamples in topology*. Dover, New York, 1995. An unabridged and unaltered republication of the work published by Springer-Verlag New York Inc., New York, 1978.
- [ST10] R. J. Smith and S. Troyanski. Renormings of  $\mathcal{C}(K)$  spaces. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM*, 104(2):375–412, 2010.
- [Tro79] S. L. Troyanski. Locally uniformly convex norms. *C. R. Acad. Bulgare Sci.*, 32(9):1167–1169, 1979. (Russian).
- [Ziz03] V. Zizler. Nonseparable Banach spaces. In *Handbook of the geometry of Banach spaces, Vol. 2*, pages 1743–1816. North-Holland, Amsterdam, 2003.

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